

Nonlinear stability and post-critical analysis of an uncertain plant with Describing Functions and Integral Quadratic Constraints

Andrea Iannelli, Andrés Marcos and Mark Lowenberg¹

Abstract—Two approaches to tackle the nonlinear robust stability problem of a plant are considered. The first employs a combination of the Describing Function method and μ analysis, while the second makes use of Integral Quadratic Constraints (IQCs). The model analyzed consists of an open-loop wing’s airfoil subject to freeplay and LTI parametric uncertainties. One of the main contributions of the work is to provide methodologies to quantitatively determine the post-critical behaviour of the system, known as Limit Cycle Oscillation (LCO). When the first approach is adopted, this is studied by means of a worst-case LCO curve, whose definition is given in the paper. The IQC framework, typically used to find asymptotic stability certificates, is applied to this scenario by introducing a restricted sector bound condition for the nonlinearity.

I. INTRODUCTION

In the last two decades great effort has been devoted in the control community to develop methodologies able to handle uncertainties and nonlinearities in a unified framework. One of the main results is represented by Integral Quadratic Constraints (IQCs) [1], a powerful tool to assess the robust stability and performance of nonlinear systems. The idea is to recast the system as a feedback interconnection of a Linear Time Invariant (LTI) plant G with an operator Δ , gathering nonlinearities and uncertainties, and describe the latter in terms of constraints on its input and output channels. IQC can be viewed as a comprehensive framework reconciling small gain [2] and positivity/passivity techniques [3].

It is possible to deal with the nonlinear robust problem within a less general framework than IQC by tackling the uncertainties and nonlinearities of the system by means of distinct tools for each. When the focus is only on LTI parameters or dynamic uncertainties, a well-established technique, which specializes the small gain theorem to the case of a structured Δ , is the structured singular value (s.s.v.) [4]. And when the focus is the nonlinearities, a way to introduce them in the frequency domain framework is represented by the Describing Functions (DF) method [5]. This technique allows to substitute the nonlinear operator with a quasi-linear one whose output is a function of some input signal features. The chief goal of this paper is to show and discuss the application of these frameworks to an engineering problem relevant to the design of modern aerospace systems, giving

novel interpretations of the obtained results and proposing their usage to study the post-critical behaviour.

A modern trend in the aeronautical industry is to design lightweight aircraft configurations to reduce fuel consumption and operating costs. The increase in flexibility is generally detrimental for the dynamic response of the aircraft, determining phenomena such as flutter, a self-excited instability featured by aeroelastic coupling between aerodynamics and structural dynamics. This problem is traditionally tackled in industry with linear nominal techniques. However, the increase in flexibility and the demand for a more realistic description of the system, compel to consider cases where these hypotheses no longer hold. The aerospace industry, for example, has recently shown interest in research aimed at evaluating the effect of the uncertainties on instabilities prompted by the control surface freeplay [6]. Among the goals of these studies, primary is the detection and characterization of Limit Cycle Oscillations (LCOs). In fact, the presence of nonlinearities leads to limited amplitude flutter, whose investigation is of well-ascertained interest in order to accomplish a satisfactory design [6].

The main contribution of this work is to propose the application of the quasi-linear robust approach (with DF and μ) and the nonlinear robust one (by means of IQC) to study this problem. A discussion on the advantages and limitations of the DF- μ approach, along with a possible interpretation of the main outcomes in terms of a worst-case LCO curve, is presented. Then, in the IQC approach, conservatism of the analyses in terms of selection of the multipliers and global validity of the results are investigated and possible strategies to address them are proposed. These two issues represent an active area of research in IQC analysis [7], [8], [9].

II. BACKGROUND

This Section presents the necessary theoretical background by means of a cursory introduction to the techniques employed. Common notation is adopted [1], [2]. The goal of the work is to study the stability of the feedback interconnection shown in Fig. 1, where G is an LTI system and Δ is a generic causal and bounded operator.

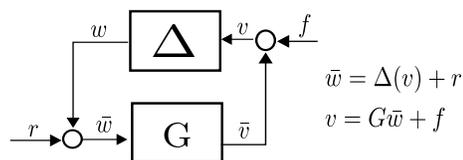


Fig. 1. Feedback interconnection

*This work has received funding from the European Union’s Horizon 2020 research and innovation programme under grant agreement No 636307, project FLEXOP.

¹ Andrea Iannelli, Andrés Marcos and Mark Lowenberg are with the Department of Aerospace Engineering, University of Bristol, BS8 1TR, United Kingdom andrea.iannelli@bristol.ac.uk/m.lowenberg@bristol.ac.uk

A. LFT modeling and μ analysis

The LFT framework provides a formal description of the feedback interconnection depicted in Fig. 1. Let $M \in \mathbb{C}^{(n+p) \times (m+q)}$ be partitioned as $M = [M_{11} \ M_{12}; M_{21} \ M_{22}]$ and $\Delta \in \mathbb{C}^{m \times n}$. The upper LFT [2] with respect to Δ is:

$$\mathcal{F}_u(M, \Delta) = M_{22} + M_{21}\Delta(I - M_{11}\Delta)^{-1}M_{12} \quad (1)$$

A crucial feature apparent in (1) is that the LFT is well posed if and only if the inverse of $(I - M_{11}\Delta)$ exists. If the operator Δ contains structured LTI uncertainties (we will indicate this by writing $\Delta = \Delta_u$), a robust stability (RS) certificate can then be obtained applying μ analysis.

The structured singular value [4] $\mu_{\Delta_u}(M_{11})$ of the complex-valued matrix M_{11} with respect to the set Δ_u is:

$$\mu = \left(\min_{\Delta_u} (\beta : \det(I - \beta M_{11}\Delta_u) = 0; \bar{\sigma}(\Delta_u) \leq 1) \right)^{-1} \quad (2)$$

where β is a real positive scalar. The result can then be interpreted as follows: if $\mu_{\Delta_u}(M_{11}) \leq 1$ then there is no perturbation matrix inside the allowable set Δ_u such that the determinant condition is satisfied, that is, the associated plant is robustly stable. On the contrary, if $\mu_{\Delta_u}(M_{11}) \geq 1$ a candidate (i.e. belonging to the allowed set) perturbation matrix exists which violates the well-posedness. It is known that $\mu_{\Delta}(M)$ is in general an NP-hard problem, thus all μ algorithms work by searching for upper and lower bounds [4].

B. Describing Function

The Describing Function method [5] aims to provide an analogous concept of frequency response for nonlinear systems. This is pursued by means of a quasi-linearization of the nonlinear operator ϕ , after the input signal form has been specified. In this work we focus on sinusoidal-input describing functions (SIDF), later abbreviated DF.

The *key* hypothesis of the DF method is that only the fundamental harmonic component has to be retained from the generical periodic output at the nonlinearity. This approximation relies on the assumption that the linear element filters out the higher harmonics. The DF of a nonlinear element with output w is thus the complex fundamental harmonic gain $N(B, \omega)$ of a nonlinearity in the presence of a driving sinusoid v of amplitude B and frequency ω :

$$N(B, \omega) = \frac{D e^{j(\omega t + \theta)}}{B e^{j\omega t}} = \frac{D}{B} e^{j\theta} = \frac{b_1 + j a_1}{B}$$

with $D(B, \omega) = \sqrt{a_1^2 + b_1^2}$; $\theta(B, \omega) = \arctan(\frac{a_1}{b_1})$

$$v = B \sin(\omega t); \quad w \simeq a_1(B, \omega) \cos(\omega t) + b_1(B, \omega) \sin(\omega t) \quad (3)$$

where a_1 and b_1 are the Fourier coefficients of the first harmonic of w . This method treats the nonlinear operator of Fig. 1 ($\Delta = \phi$ when it only gathers nonlinearities) in the presence of a sinusoid inputs as if it were a linear element with a frequency response $N(B, \omega)$. Linear theory is then applied to the quasi-linearized system, searching for points of neutral stability which are interpreted as LCOs in the nonlinear system.

C. Integral Quadratic Constraints

IQC is a well established technique to deal with stability and performance analysis of nonlinear and uncertain systems [1] in a unified framework. Let $\Pi : j\mathbb{R} \rightarrow \mathbb{C}^{(n+m) \times (n+m)}$ be a measurable Hermitian-valued function, named multiplier. Two signals $v \in \mathcal{L}_2^n[0, \infty]$ and $w \in \mathcal{L}_2^m[0, \infty]$ (with Fourier transforms \hat{v} and \hat{w}) satisfy the IQC defined by Π if:

$$\int_{-\infty}^{+\infty} \begin{bmatrix} \hat{v}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \hat{v}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix} d\omega \geq 0 \quad (4)$$

A bounded and causal operator Δ satisfies the IQC defined by Π if (4) holds for all v and $w = \Delta(v)$. Conditions for *absolute stability* of the feedback interconnection of G and Δ are given in terms of matrix inequalities to be verified over the entire frequency spectrum [1]. In order to facilitate the numerical solution of this problem, it is common practice to factorize the multiplier Π as $\Psi^* S \Psi$ where $S = S^T$ is a real matrix variable and Ψ is a transfer matrix constructed from pre-selected basis transfer functions. The search for stability certificates can then be recast into a Linear Matrix Inequality (LMI) problem. In particular, stability is guaranteed if there exists a matrix $P = P^T$ such that:

$$\begin{bmatrix} \check{A}^T P + P \check{A} & P \check{B} \\ \check{B}^T P & 0 \end{bmatrix} + \begin{bmatrix} \check{C}^T \\ \check{D}^T \end{bmatrix} S \begin{bmatrix} \check{C} & \check{D} \end{bmatrix} < 0 \quad (5)$$

with $[\check{A}, \check{B}, \check{C}, \check{D}]$ obtained from the state-space realizations of G and Ψ . The core effort to reduce the conservatism associated to the results consists in finding suitable multipliers Π_i describing the input/output relation of the operator Δ .

III. PROBLEM STATEMENT

In the past two decades it has been clearly asserted the need to take into account the effects of nonlinearities and uncertainties [10] when studying aeroelastic phenomena. In particular, LCOs must be avoided in mechanical systems since they are likely to degrade fatigue life and provoke critical damages. Aircraft design requirements (for both civil and military aviation) formulate constraints on LCO accelerations in prescribed points of the airframe [6]. These quantities can be estimated provided that a characterization of the LCO in terms of *amplitude* and *frequency* is available, hence motivating the focus of this work.

In the first Section an overview of the model employed to analyze the benchmark is presented. The interested reader is referred to [11] for a detailed discussion on the aeroelastic modeling aspects. The other Sections describe how uncertainties and nonlinearities are treated in the pursued approaches.

A. Aeroelastic model

The aeroelastic system consists of a rigid airfoil with lumped springs simulating the 3 degrees of freedom (DOFs): plunge h , pitch α and trailing edge flap β . Theodorsen's unsteady formulation is employed to model the aerodynamics. If $X = [h \ \alpha \ \beta]^T$ and $L = [-L_h \ M_\alpha \ M_\beta]^T$ are defined as the vectors of the degrees of freedom and

corresponding aerodynamic loads respectively, the aerodynamic model provides the generalized Aerodynamic Influence Coefficient (AIC) matrix A_g , where $A_{g(ij)}$ represents the transfer function from the degree of freedom j in X to the aerodynamic load component i in L . Since the AIC matrix has a non-rational dependence on the Laplace variable s , a rational approximation of A_g by means of the Minimum State method [11] is employed, which enables to obtain a state-space description of the dynamics:

$$\dot{x} = \begin{bmatrix} \dot{x}_s \\ \dot{x}_a \end{bmatrix} = \begin{bmatrix} A_{ss} & A_{sa} \\ A_{as} & A_{aa} \end{bmatrix} \begin{bmatrix} x_s \\ x_a \end{bmatrix} = Ax \quad (6)$$

where A is the state-matrix (function of the airspeed V), x is the vector of states and x_s and x_a are respectively the structural and aerodynamic states, the latter needed for the rational approximation of the unsteady operator. The total size of the plant in our example is 9 (6 structural and 3 aerodynamic states).

Flutter analysis evaluates the conditions at which the dynamic aeroelastic system (6) loses its stability. The result is the prediction of the so-called flutter speed V_f , below which the system is guaranteed to be stable.

B. Model uncertainties

Parametric uncertainties are used to describe parameters whose values are not known with a satisfactory level of confidence. Considering a generic uncertain parameter d , with λ_d indicating the uncertainty level with respect to a nominal value d_0 , a general representation is given by:

$$d = d_0 + \lambda_d \delta_d \quad (7)$$

where $\|\delta_d\| \leq 1$. This study will take into account a 10% uncertainty in the following parameters: K_h , and K_α (bending and torsional stiffness); static moment of the airfoil S_α ; I_α , and I_β (airfoil and flap moment of inertia). As explained later in Sec. III-C, the flap stiffness K_β , affected by freeplay nonlinearity but not uncertainty, will also be handled within the LFT framework.

The LFT paradigm enables to manipulate the nominal system by simply introducing the expression (7), specialized for each uncertain parameter, into the state-matrix (6) and using well-established realization techniques (in this work by means of the LFR toolbox [12]) to obtain the corresponding upper LFT (1). Particularly relevant is the state-space realization of the transfer matrix $M_{11}(s)$:

$$M_{11}(s) = C_G(sI_n - A_G)^{-1}B_G + D_G \quad (8)$$

where $A_G = A$ from (6). The subscript G is to remark that M_{11} coincides with the plant G in Fig. 1. This representation of the uncertain plant is the starting point for the study of the system robust stability with either μ or IQC analysis.

μ analysis can be straightforwardly applied to evaluate the robustness of the system. Once the LFT is built up, calculating the matrix M_{11} in (2) basically amounts to evaluating (8) at $s = j\omega$, where ω belongs to the set of frequencies employed in the analyses. The Robust Control Toolbox in MATLAB [4] will be adopted here.

IQC analysis requires to characterize Δ_u in terms of a multiplier Π satisfying (4). It is well-known that for constant real scalar uncertainties $\delta_d \leq 1$, a candidate is:

$$\Pi^R = \begin{bmatrix} X(j\omega) & Y(j\omega) \\ Y(j\omega)^* & -X(j\omega) \end{bmatrix} \quad (9)$$

where $X(j\omega) = X(j\omega)^* \geq 0$ and $Y(j\omega) = -Y(j\omega)^*$ are generic bounded and measurable matrix functions (named D - G scalings in robust control theory) [1].

The IQC β toolbox [13], employed in this work for the analyses, allows to formulate the problem declaring the connections among the signals of the system by *linking* them through appropriate sub-functions. The LTI system G gives, by means of the state-space realization (8), the transfer matrix from w and v . The relation between v and w can instead be defined by means of the sub-function `iqc_ltigain`, which implements a parametrization of the multiplier (9) with first order low-pass filters with poles a_f .

C. Freeplay nonlinearity

Freeplay, also called dead-zone, often arises in mechanical and electrical systems where the first part of the input is needed to overcome an initial opposition at the output. The freeplay nonlinearity for the system is concentrated in the control surface stiffness K_β . The mathematical expression for the elastic moment M_β^E can be written as:

$$M_\beta^E = \begin{cases} K_\beta^L(\beta - \bar{\delta}); & |\beta| > \bar{\delta} \\ 0; & |\beta| < \bar{\delta} \end{cases} \quad (10)$$

where $\bar{\delta}$ is defined as the (positive) freeplay size and K_β^L is the flap stiffness in the linear case ($\bar{\delta}=0$). It is worth noting some important properties of this nonlinearity: it is odd (i.e. the relation is symmetric about the origin), memoryless (i.e. only one output is possible for any given value of the input), and static (i.e. no dependence upon the input derivatives).

The DF function N^F associated to freeplay can be obtained analytically [5] and is here normalized such that $0 < N^F < 1$. Due to the aforementioned properties held by freeplay, its describing function is a pure real gain (i.e. $\theta = 0$) not depending on frequency, but only on the amplitude of the input signal B (here specified as β_s), in particular on its ratio with $\bar{\delta}$. The application of DF enables to give an expression for the elastic moment M_β^E in (10) as:

$$M_\beta^E = K_\beta^{QL} \beta; \quad \text{with} \quad K_\beta^{QL} = N^F(\beta_s) K_\beta^L \quad (11)$$

where K_β^{QL} is the quasi-linear flap stiffness. The flutter speed V_f , obtained from an eigenvalue analysis of (6), is thus associated to an LCO of amplitude β_s and frequency ω_s equal to the imaginary part of the unstable eigenvalue.

When IQC analysis is pursued, a set of multipliers, describing different properties of the nonlinearity, can be selected. In this work we will consider: sector multiplier Π^S with bounds $[\alpha, \eta]$; Popov multiplier Π^P reflecting the time-invariance; Zames-Falb multiplier Π^{ZF} for monotonic and odd nonlinearities, whose slope is restricted in the sector $[\alpha_1, \eta_1]$. For the latter, the user must define, other than the

sector bounds, also the length N_H and the pole location a_H of the multiplier parametrization.

To conclude the IQC description of freeplay, it is proposed here to compute the plant G by building an LFT of the nonlinear stiffness K_β , that is, treating it as if it was an uncertain parameter. Once the sector bound $[\alpha, \eta]$ is specified, a range of variation $K_{\beta-1} < K_\beta < K_{\beta-2}$ is defined, with $K_{\beta-1} = 2\alpha - \eta$ and $K_{\beta-2} = \eta$. In this way, after the range normalization $\|\delta_{K_\beta}\| \leq 1$, the sector $[0,1]$ automatically holds for Π^S (note that $\delta_{K_\beta} = 0$ corresponds to $K_\beta = \alpha$ and $\delta_{K_\beta} = 1$ corresponds to $K_\beta = \eta$). With this implementation, the nonlinear uncertain system can be manipulated efficiently [12] within a unified framework.

IV. QUASILINEAR ROBUST ANALYSIS WITH μ -DF

This Section shows the results obtained from the application of the DF- μ approach to the study of LCOs in the wing section affected by freeplay and uncertainties. The values of the parameters defining the nominal model are reported in [14], where the benchmark study is presented. If no uncertainties are included in the model, the strategy outlined in Sec. III-C can be employed. Fig. 2 showcases the values of flutter speed V_f and associated frequency ω_s corresponding to a variation of flap stiffness between 0 and the linear value $K_\beta^L = 3.9$ N (that is, as the associated describing function N^F varies from 0 and 1).

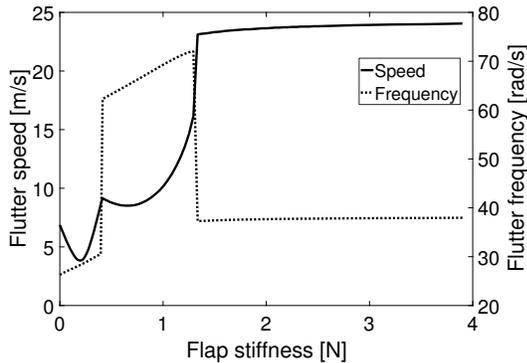


Fig. 2. Flutter speed and frequency vs. flap stiffness

Due to the existing relation between K_β^{QL} and β_s (11), these results can be shown in a plot with airspeed versus oscillation amplitude, see Fig. 3. The DF method is instrumental in guaranteeing this connection and enabling to transfer the information coming from multiple linear flutter analyses to an LCO characterization [15]. Stable and unstable oscillations are depicted respectively with solid and dashed lines, according to the criterion in [16]. Four regions can be identified as the airspeed increases: (i) $V < V_0 (= 3.8 \frac{m}{s})$, where the system is stable; (ii) $V_0 < V < V_1 (= 9 \frac{m}{s})$, where the system undergoes LCOs associated with the plunge instability (with amplitude given by the upper stable branch); (iii) $V_1 < V < V_2 (= 23.2 \frac{m}{s})$ where the LCO switches to the pitch instability (the frequency correspondingly changes, as in Fig. 2) and the amplitude visibly increases; and (iv), for speeds greater than V_2 where there is an asymptote.

If the nonlinear system is also affected by uncertainties, it is of interest to estimate how the stability properties (in terms

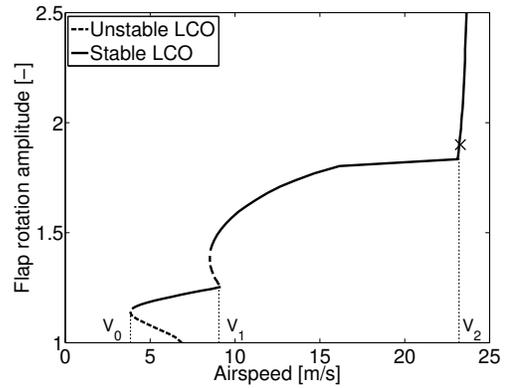


Fig. 3. Flap rotation LCO amplitude $\frac{\beta_s}{\delta}$ against airspeed (nominal system)

of V_0), and the LCO features (represented by amplitude β_s , frequency ω_s and the other characteristic speeds) vary due to the terms in Δ_u . This task can be approached as follows: for a given amplitude β_s , the block ϕ has a fixed value and the LFT only consists of uncertain parameter ($\Delta = \Delta_u$); the problem can then be formulated as a standard RS calculation with μ , looking at the smallest airspeed at which the system is robustly unstable (i.e. the smallest V such that $\mu = 1$) and at the related peak frequency. This procedure leads to what is named here as the *worst-case LCO curve*, i.e. the equivalent of Fig. 3 but where a measure of the LCO properties degradation in the face of the uncertainties is provided. To this end, a flap stiffness gridding is calculated and, at each point, a bisection-like algorithm searching for the airspeed V which attains first the RS violation condition is implemented. Two curves are thus obtained: one for $\mu_{UB} = 1$ (dashed) and one for $\mu_{LB} = 1$ (dotted). The results, with the nominal case (solid), are shown in Fig. 4.

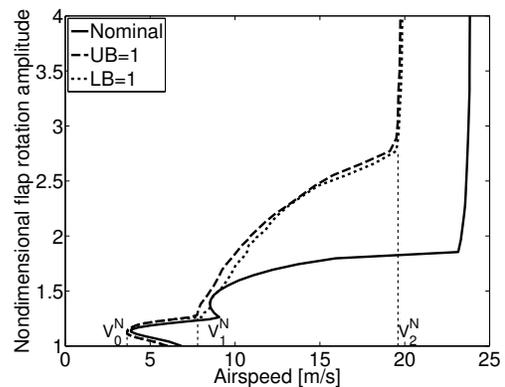


Fig. 4. LCO amplitude $\frac{\beta_s}{\delta}$ against airspeed– a worst-case LCO curve

This plot can be interpreted as a worst-case analysis of the nonlinear flutter problem in terms of LCO onset and amplitude. In fact, it assesses how the properties discussed before (with reference to Fig. 3) degrade. The first information that can be inferred is the smallest airspeed for which the system experiences LCO. The assumed set of uncertainties slightly decreases this value from $V_0 = 3.8 \frac{m}{s}$ to $V_0^N = 3.6 \frac{m}{s}$. As the airspeed is increased, the regions highlighted in Fig. 3 are still detectable but V_1 and V_2 are also shifted towards smaller values (V_1^N and V_2^N). Furthermore, the plot also allows to

clearly appreciate, especially in the third region, a sizable growth in amplitude. Although not reported in the plot, at each LCO point it is possible also to associate the oscillation frequency (corresponding to the peak value of μ).

V. NONLINEAR ROBUST ANALYSIS WITH IQCS

This Section presents the application of IQC analysis to the studied test case. It is known that a possible drawback of this approach lies in the conservatism associated to the results. This work investigates two related aspects: the selection of the multipliers and the local/global validity of the results.

A. Sensitivity to multipliers' selection and parametrization

The first analysis considers the nominal airfoil affected by freeplay nonlinearities. Once the IQC description of the freeplay nonlinearity is provided as documented in Sec. III-C, the airspeed is increased until the LMI problem in (5) becomes unfeasible (the first airspeed for which this happens is referred to as V_{unf}). In these first analyses it is assumed that the nonlinearity is in the sector $[0, K_\beta^L]$. In Tab. I the analyses performed are shown reporting for each test the multiplier (with parametrization if any), the number of decision variables, and the airspeed V_{unf} .

TABLE I
IQC ANALYSIS OF THE NONLINEAR NOMINAL SYSTEM

Multiplier & Options	Size	V_{unf}
Π^S	47	-
Π^S, Π^P	48	$3.81 \frac{m}{s}$
$\Pi^S, \Pi^P, \Pi^{ZF}([1, 1 \frac{rad}{s}])$	80	$3.82 \frac{m}{s}$

When only the sector bound condition is enforced, no feasible solution is achieved. The Popov multiplier Π^P , encompassing the time invariance of the freeplay, is then added and this enables to find $V_{unf} = 3.8 \frac{m}{s}$, confirmed also via Π^{ZF} . This value is the same as the airspeed V_0 detected in Fig. 3, and thus indicates that the approach of considering the entire sector $[0, K_\beta^L]$ is equivalent to look for the smallest airspeed such that the system experiences an LCO. This is in line with the standard application of IQC which looks for asymptotic stability certificates of the analyzed system.

Next, the system is considered to be also affected by uncertainties. The multiplier Π^R is employed to model the uncertainties channels in Δ and the obtained results are presented in Tab. II. It is clear from the results that when only one filter is used for Π^R , the algorithm is not able to find a feasible solution. In fact, it is decisive to increase the number of filters in order to match the largest stable airspeed $V_{unf} = 3.6 \frac{m}{s}$ found in the analyses of Fig. 4. Note that the filters frequencies are chosen close to the expected flutter frequencies (available from the analyses in Sec. IV).

The results presented confirm the well-known dependence of IQC analysis on multipliers selection and parametrization. However, in this study it is stressed the importance of having reference results (here provided by the DF- μ approach). Firstly, they provide a measure of the conservatism associated to the infeasibility of the LMI problem and therefore may point out the need to employ a more refined set of

TABLE II
IQC ANALYSIS OF THE NONLINEAR UNCERTAIN SYSTEM

Multiplier & Options	Size	V_{unf}
$\Pi^R (40 \frac{rad}{s}), \Pi^S, \Pi^P, \Pi^{ZF}([1, 40 \frac{rad}{s}])$	390	-
$\Pi^R (40 \frac{rad}{s}, 80 \frac{rad}{s}), \Pi^S, \Pi^P, \Pi^{ZF}([1, 40 \frac{rad}{s}])$	890	$3.1 \frac{m}{s}$
$\Pi^R (1 \frac{rad}{s}, 40 \frac{rad}{s}, 80 \frac{rad}{s}), \Pi^S, \Pi^P, \Pi^{ZF}([1, 40 \frac{rad}{s}])$	1590	$3.6 \frac{m}{s}$

multipliers. While this is typically accomplished with a frequency sweep of the filter poles (time-consuming and not always successful), the availability of auxiliary reference results can inform the improvement of the parametrization for the multipliers (e.g. characterizing the sensitivity of the instability to the blocks Δ_i and therefore focusing only on the refinement of the associated multipliers Π_i ; highlighting critical frequencies of the systems). As for the latter aspect, the values of the filter poles are selected here considering the expected unstable frequencies of the systems obtained by DF-nominal analysis or DF- μ approach.

B. Local analysis

The analyses presented so far consider the standard sector definition for the freeplay. The certificates found with this approach guarantee asymptotic stability of the system. In fact, only the largest airspeed at which the system settles down to the original equilibrium when subject to any vanishing perturbation can be inferred. It is well understood in the literature [1], [7], [9] that results holding *locally* can greatly improve the analysis of systems via IQC. Therefore now the IQC description of the nonlinearity is relaxed by considering a local sector constraint definition. In Fig. 5 the freeplay nonlinearity is depicted with dashed line, whereas the proposed reduced sector consists of the grey area.

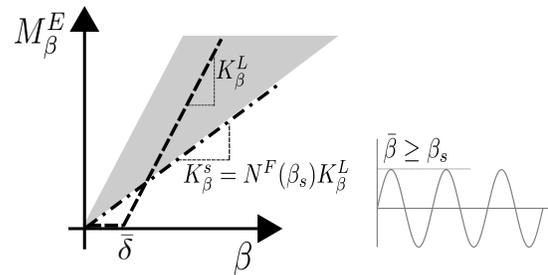


Fig. 5. Reduced sector constraint (grey) for freeplay nonlinearity (dashed). The premise of this relaxation is that the DF method provides, for a given freeplay size $\bar{\delta}$, a relation between the amplitude of the nonlinear response β_s and the equivalent stiffness associated to the freeplay K_β^{QL} (11). If a lower bound on β_s is assumed, then $K_\beta^{QL}(\beta_s)$ can be taken as the lower limit of the sector K_β^s (dashed-dotted line). The bound on β_s can be interpreted as an oscillation level that can be withstood by the structure, and thus it is tolerated as post-critical response. IQC will then allow to determine the largest airspeed V_{unf} such that the system does not experience any oscillatory motion of amplitude greater than β_s . The local characteristic of the analysis is thus obtained by looking

at the post-critical behaviour of the system. In particular, it prescribes to detect only unstable responses featured by a minimum level of amplitude. In view of the importance of the LCO amplitude characterization for the design of aerospace structures (as remarked in the previous Sections), this is believed to be a profitable tool for flutter analysis. Although IQC-based synthesis is a non-convex problem and thus still an active area of research, this formulation could also be exploited to design feedback control laws for active reduction of LCO amplitude.

Table III shows the results obtained applying this approach. The upper sector limit $\eta = \eta_1$ is fixed at K_β^L as in the previous analyses. A different lower limit for the sector $\alpha = \alpha_1 = K_\beta^s$, with an associated smallest amplitude $\frac{\beta_s}{\delta}$, is instead selected for each test and the smallest unfeasible airspeeds (namely V_{unf}^1 for the nonlinear nominal system and V_{unf}^2 for the nonlinear uncertain one) are reported. This analysis, repeated on a grid of values of K_β^s , can be interpreted as a nominal and robust characterization of the nonlinear response of the system in that it provides the highest airspeed at which the system can be operated if oscillations below a certain threshold $\frac{\beta_s}{\delta}$ are tolerated.

TABLE III
LOCAL IQC ANALYSIS

$K_\beta^s [N]$	$\frac{\beta_s}{\delta}$	$V_{unf}^1 [\frac{m}{s}]$	$V_{unf}^2 [\frac{m}{s}]$
0.32	1.21	5.9	4.3
0.86	1.5	9.0	8.45
1.15	1.69	12.2	9.2
1.39	1.9	23.05	10.3
1.72	2.2	23.5	12.3
2.00	2.5	23.6	15.7

Looking at Tab. III it can be noted that different speeds are predicted for each lower sector bound. For $\frac{\beta_s}{\delta} \leq 1.5$, the degradation due to the uncertainties, measured by the difference between V_{unf}^1 and V_{unf}^2 , is not remarkable. As the amplitude is increased (note that a bold line is employed in Tab. III to emphasize the two regions), it is evident a greater effect of the uncertainties in worsening the response. For example, assume a nominal analysis cleared the system, able to withstand oscillations of amplitude up to $1.9\bar{\delta}$, to operate at $V=16\frac{m}{s}$. The latter amplitude takes place in nominal conditions at $V_{unf}^1=23.05\frac{m}{s}$ (see Tab. III). The proposed analysis then reveals that in the face of uncertainties the system could exhibit an LCO greater than $2.5\bar{\delta}$ at $V_{unf}^2 = 15.7\frac{m}{s}$, which is actually slightly less than the cleared nominal airspeed- with the risks this represents.

The trend in Tab. III is in good agreement not only qualitatively, but also quantitatively, with the worst-case LCO plot in Fig. 4. The intersections of horizontal lines (drawn for different values of the ordinate $\frac{\beta_s}{\delta}$) with the nominal and robust curves give points having as x coordinate approximately the corresponding values of V_{unf}^1 and V_{unf}^2 .

It is important to note finally that the accuracy of the conclusions inferred with this *local* approach is expected to depend in general on the applicability of the hypotheses underlying DF theory to the particular system examined

(these were commented on for this problem in [15]).

VI. CONCLUSION

This paper studied the stability and post-critical behaviour of an airfoil subject to freeplay and LTI parametric uncertainties. Two approaches were presented, the first featured by a combination of Describing Function and μ analysis, and the second based on Integral Quadratic Constraints.

When the DF- μ approach was adopted, the conditions at which the system lost stability could be determined, and changes in the nonlinear response with respect to the nominal case, estimated. A methodology to build what was named here a worst-case LCO curve was proposed.

In pursuing the IQC approach, emphasis was given to the conservatism of the analyses. In a first step, this was ascribed to the parametrization of the multipliers and a sensitivity of the results was proposed. Then, the global and local nature of the stability certificates obtained via IQC was considered. It was proposed, based on a connection with the DF theory, a restricted sector bound condition allowing to detect nonlinear responses above a certain amplitude threshold. The prowess of this description, believed to have practical consequences for nonlinear stability analysis and control synthesis, was validated with the outcomes of the DF- μ approach.

REFERENCES

- [1] A. Megretski and A. Rantzer, "System analysis via integral quadratic constraints," *IEEE Transactions on Automatic Control*, vol. 42, no. 6, pp. 819–830, Jun 1997.
- [2] K. Zhou, J. C. Doyle, and K. Glover, *Robust and Optimal Control*. Prentice-Hall, Inc., 1996.
- [3] C. Desoer and M. Vidyasagar, *Feedback Systems: Input-output Properties*, ser. Classics in Applied Mathematics. Society for Industrial and Applied Mathematics, 1975.
- [4] G. Balas, R. Chiang, A. Packard, and M. Safonov, *Robust Control Toolbox*, 2009.
- [5] A. Gelb and W. E. Vander Velde, *Multiple-Input Describing Functions and Nonlinear System Design*. McGraw-Hill, 1968.
- [6] P. Bansal and D. Pitt, "Uncertainties in control surface free-play and structural properties and their effect on flutter and LCO." AIAA Scitech Conference, January 2014.
- [7] J. Veenman, C. W. Scherer, and H. Koroğlu, "Robust stability and performance analysis based on integral quadratic constraints," *European Journal of Control*, vol. 31, pp. 1 – 32, 2016.
- [8] J. Carrasco, W. P. Heath, and A. Lanzon, "Equivalence between classes of multipliers for slope-restricted nonlinearities," *Automatica*, vol. 49, no. 6, pp. 1732 – 1740, 2013.
- [9] A. Knobloch, H. Pfifer, and P. Seiler, "Worst Case Analysis of a Saturated Gust Loads Alleviation System." AIAA Scitech, 2015.
- [10] C. L. Pettit, "Uncertainty quantification in aeroelasticity: Recent results and research challenges," *Journal of Aircraft*, vol. 41, no. 5, 2004.
- [11] A. Iannelli, A. Marcos, and M. Lowenberg, "Aeroelastic modeling and stability analysis: A robust approach to the flutter problem," *International Journal of Robust and Nonlinear Control*, 2017.
- [12] J. Magni, "Linear fractional representation toolbox modelling, order reduction, gain scheduling." DCSO, ONERA, Systems Control and Flight Dynamics Department, Technical Report TR 6/08162, 2004.
- [13] U. Jonsson, C.-Y. Kao, A. Megretski, and A. Rantzer, *A Guide to IQC β : A MATLAB Toolbox for Robust Stability and Performance Analysis*.
- [14] D. Tang, E. Dowell, and L. Virgin, "Limit Cycle Behavior of an Airfoil with a Control Surface," *Journal of Fluids and Structures*, vol. 12, no. 7, pp. 839 – 858, 1998.
- [15] A. Iannelli, A. Marcos, and M. Lowenberg, "Limit Cycle Oscillation amplitude tailoring based on Describing Functions and μ Analysis," in *Advances in Aerospace Guidance, Navigation and Control*, 2017.
- [16] M. Anderson, "Pilot-induced oscillations involving multiple nonlinearities," *J. of Guidance, Control and Dynamics*, vol. Vol. 21, no. 5, pp. 786–791, 1998.