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Linear Parameter Varying Modeling of the Boeing 747-100/200 Longitudinal Motion

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This paper presents a Linear Parameter Varying (LPV) modeling approach for the longitudinal motion of a Boeing 747 series 100/200. The three approaches used to obtain the quasi-LPV models are Jacobian Linearisation, State Transformation, and Function Substitution. Linear Parameter Varying models are a key step in applying LPV control synthesis which guarantee a level of stability and robustness for the closed loop. The models are obtained for the Up-and-Away flight envelope of the Boeing 747-100/200. Comparisons of the three models in terms of their advantages, drawbacks and modeling difficulty are presented. Open loop time responses show all the three quasi-LPV models matching the behavior of the nonlinear model when in the trim region. The Function Substitution quasi-LPV model in this particular case is the only valid model for the full flight envelope of interest.

1 Nomenclature

$\bar{c}$	-	wing chord, meter.
$c_1, \ldots c_9$	-	inertia coefficients.
$\bar{q}$	-	dynamic pressure, N/m^2.
S	-	reference surface area, m^2.
z_{eng}	-	engine position z-axis, m.
T, Tn	-	thrust force, Newton.
V_{TAS}	-	true airspeed, m/sec.
α	-	angle of attack (AoA), deg.
α_w	-	wing design plane $\alpha_w = \alpha + 2$.
s_α, s_β	-	sine - AoA & sideslip
c_α, c_β	-	cosine - AoA & sideslip
$c.g.$	-	center of gravity
$f.r.l.$	-	fuselage/flight reference line

2 Introduction

In recent years an emerging approach to control theory, Linear Parameter Varying (LPV) control, has tried to establish itself as a reliable alternative to classical gain-scheduling. Gain scheduling is a standard method to design controllers for dynamical systems over a wide performance envelope. It yields a global controller based on interpolation of a family of locally linearized controllers. Some drawbacks of this methodology are its *ad hoc* character and more important, the fact that the controller obtained comes with no guarantees on its stability or robustness other than at the design points (this is specially critic for rapid variations in the scheduling parameters). Linear Parameter Varying control synthesis naturally fits into the gain scheduling framework, while imbuing it with stability and robustness assurances. LPV control synthesis techniques have already been used, with varying levels of success, for a wide array of dynamical systems. These include high performance aircraft as representative as the F-14[1] , F-16[2] , F-18[3] and the VAAC Harrier[4] , turbofan engines[5,6] and missiles[7,8] . A condition to apply LPV control synthesis is to transform the nonlinear model of the system into an LPV model, hence LPV modeling becomes a key issue in the design of LPV controllers[4,9,10] .

Generally, control designers will use a family of linear, time-invariant (LTI) plants at different points of interest throughout the flight envelope in order to obtain the LPV plant[1,3,5,6] . In references[4,9,10] State Transformations techniques were used to derive a reliable LPV model. The Function Substitution LPV modeling approach was used in[2,11] to model the F-16 dynamics. In the present paper the three different approaches are used to model the Boeing 747-100/200 longitudinal motion for the Up-and-Away flight envelope. A discussion of their respective advantages and disadvantages together with our observations with respect to the difficulty and convenience of each modeling technique is presented in the conclusions. In order to validate the LPV models open-loop time responses are obtained and then compared to the transient response of the nonlinear model. The longitudinal nonlinear model of the Boeing 747-100/200 is simplified by reducing the complexity

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of the aerodynamic coefficients while still maintaining a high degree of accuracy with respect to the full set of aerodynamic coefficients. This reduction is carried out through analytical and simulation studies.

The results show that for a certain region in the flight envelope studied, i.e. the acceptable trim region, all the three quasi-LPV (qLPV) models are valid. The first two approaches: Jacobian Linearisation, and the State Transformation depend on trim points or trim functions which then limit their existence to the aforementioned feasible trim-map. The Function Substitution LPV model does depend only on a single trim point and therefore its valid domain, in the sense of flight envelope covered, is not hampered by the trim region. Therefore, only the Function Substitution quasi-LPV model matches the nonlinear response of the Boeing 747 for deflections that lead the aircraft outside of the valid trim envelope.

The outline of this paper is as follows. The next section provides a theoretical background of LPV and qLPV systems (these two terms will be used freely throughout the paper since the ideas for the former can be applied to the latter) and of the three modeling approaches. In Section 4, the nonlinear model for the longitudinal motion of the Boeing 747 is presented. The software used in this research is presented in Section 5. Section 6 presents the qLPV models obtained and discusses their differences. The results of this work are summarized in the final section.

3 Theory

We start by introducing the idea and the formal definitions of an LPV and a quasi-LPV system. The class of finite dimensional linear systems whose state-space entries depend continuously on a time-varying parameter vector, ρ(t), is called Linear Parameter Varying, LPV. The trajectory of the vector valued signal, ρ(t), is assumed not to be known in advanced, although its value can be accessible (measured) in real time and is constrained *a priori* to lie in a specified bounded set. The idea behind using LPV systems *in lieu* of Linear Time Invariant, LTI, or Linear Time Varying, LTV, is to take advantage of causal knowledge of the dynamics of the system. In the LPV framework, this causal relationship between the vector value signal, ρ(t) and the plant allows the control designer to restrict the dependence of the controller dynamics to variations in the plant's characteristics, taking full advantage of the information provided by the scheduling variables. The formal definition of an LPV systems is given below.

Definition 3.1 (Linear Parameter Varying Systems[12]) *Given a compact subset $\mathcal{P} \subset \mathcal{R}^s$, the parameter variation set $\mathcal{F}_{\mathcal{P}}$ denotes the set of all piecewise continuous functions mapping $\mathcal{R}^+$ (time) into* ρ *with a finite number of discontinuities in any interval. And given continuous functions: $A:\mathcal{R}^s \to \mathcal{R}^{n\times n}$, $B:\mathcal{R}^s \to \mathcal{R}^{n\times n_u}$, $C:\mathcal{R}^s \to \mathcal{R}^{n_y\times n}$, and $D:\mathcal{R}^s \to \mathcal{R}^{n_y\times n_u}$.*

A nth order linear parameter-varying system is defined as:

$$\begin{bmatrix}\dot{x}(t)\\ y(t)\end{bmatrix} = \begin{bmatrix}A(\rho(t)) & B(\rho(t))\\ C(\rho(t)) & D(\rho(t))\end{bmatrix}\begin{bmatrix}x(t)\\ u(t)\end{bmatrix} \tag{1}$$

where $\rho \in \mathcal{F}_{\mathcal{P}}$.

Quasi-LPV systems arise whenever any of the scheduling variables, $\rho(t)$, is also a state of the system. By treating the scheduling parameters as independents, the techniques used to design LPV controllers, K_ρ, can be applied[7] .

Definition 3.2 (Quasi Linear Parameter Varying Systems) *Given a Linear Parameter Varying system as defined in (Definition 3.1) a Quasi-Linear Parameter Varying system is obtained if the state vector, x(t) can be decomposed into scheduling states, $z(t) \in \mathcal{F}_{\mathcal{P}}$ and non-scheduling states, w(t).*

$$x(t) = [z(t)\;\; w(t)]^T \tag{2}$$

Thus, the Quasi-LPV model is defined by:

$$\begin{bmatrix}\dot{z}(t)\\ \dot{w}(t)\\ y(t)\end{bmatrix} = \begin{bmatrix}A_{11}(\rho(t)) & A_{12}(\rho(t)) & B_1(\rho(t))\\ A_{21}(\rho(t)) & A_{22}(\rho(t)) & B_2(\rho(t))\\ C_1(\rho(t)) & C_2(\rho(t)) & D(\rho(t))\end{bmatrix}\begin{bmatrix}z(t)\\ w(t)\\ u(t)\end{bmatrix} \tag{3}$$

where the scheduling parameter vector is $\rho(t) = [z(t)\;\; \Omega(t)]$, and $\Omega(t) \in \mathcal{R}^{n_p}$ are exogenous scheduling variables.

The selection of the adequate scheduling variables that capture the nonlinearities of the system is a task that is not always obvious *a priori*. There are three approaches, to the best of our knowledge, that can be used to obtain a reliable LPV/quasi-LPV model. They are Jacobian Linearisation, State Transformation and Function Substitution. Assume that the nonlinear model is of the following class

$$\begin{bmatrix}\dot{z}(t)\\ \dot{w}(t)\end{bmatrix} = \begin{bmatrix}k_1(\rho(t))\\ k_2(\rho(t))\end{bmatrix} + \begin{bmatrix}A_{11}(\rho(t)) & A_{12}(\rho(t))\\ A_{21}(\rho(t)) & A_{22}(\rho(t))\end{bmatrix}\begin{bmatrix}z(t)\\ w(t)\end{bmatrix} + \begin{bmatrix}B_1(\rho(t))\\ B_2(\rho(t))\end{bmatrix}\begin{bmatrix}u(t)\end{bmatrix} \tag{4}$$

$$y(t) = [z(t)\; w(t)] \tag{5}$$

where $z(t) \in \mathcal{R}^{n_z}$ is the scheduling-states vector, $w(t) \in \mathcal{R}^{n_w}$ the non-scheduling states, $u(t) \in \mathcal{R}^{n_u}$ is the control input vector, and the measured output vector is given by y. The A, B, and k matrices can be nonlinear in the scheduling vector, $\rho(t)$. Without loss of generality assume that there are no exogenous scheduling variables ($\rho(t) = z(t)$). From this type of system it is possible to develop each of the qLPV models. The dependency on time will be dropped from now on.

The Jacobian Linearisation approach is the most widespread methodology to linearize nonlinear systems. It can be used to create a qLPV or family of qLPV models with respect to a set of equilibrium points that represents the flight envelope of interest. The resulting model is an approximation to the dynamics of the nonlinear plant around this set of equilibrium points. Since it is a first order approximation it could lead to divergent behavior, with respect to the nonlinear model, for large control inputs. It is generally impossible to capture the transient behavior of the nonlinear plant by this method. For certain class of nonlinear systems it is possible to account for the essential features of the transient response[10] . The main concept of this method is to use first order Taylor's expansion of the nonlinear model (4) with respect to a trim point. Then rewrite the resulting equations for the different states in a state-space form. It is easy to see that the trim values, and all the elements in the state-space matrices depend on the scheduling variables and hence the model is quasi-LPV. A detailed theoretical derivation of a Jacobian model is given in[13] .

The second approach is called State Transformation since the quasi-LPV model is obtained through exact transformations of the nonlinear states. This technique was introduced by Shamma and Cloutier[9] , and it has been applied to a wide range of applications[2,4,7,10] . In order to use this technique it is necessary to have the special class of nonlinear systems given above (4). It is required that $n_z = n_u$, otherwise the system is not detectable (something to account for later on when synthesizing an LPV controller). If the system meets these requirements, it can be transformed into a qLPV model whose state-space data is a function of the scheduling variables, ρ(t). This implies the scheduling parameters must be available in real-time for measurement.

Assume there exist continuously differentiable functions $w_{eq}(\rho(t))$ and $u_{eq}(\rho(t))$ such that for every $\rho(t)$ the system is in steady state

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} k_1(\rho) \\ k_2(\rho) \end{bmatrix} + \begin{bmatrix} A_{11}(\rho) & A_{12}(\rho) \\ A_{21}(\rho) & A_{22}(\rho) \end{bmatrix} \begin{bmatrix} z \\ w_{eq}(\rho) \end{bmatrix} + \begin{bmatrix} B_1(\rho) \\ B_2(\rho) \end{bmatrix} \begin{bmatrix} u_{eq}(\rho) \end{bmatrix} \tag{6}$$

One of the main drawbacks of this method is that there is no assurance of the existence of trim values for the entire flight envelope of interest for a particular combination of the scheduling variables. It is only possible to assure that the model obtained is valid in the "restricted" envelope (that with acceptable trim functions). Consequently, before generating the qLPV model it is necessary to investigate the realizable trim map.

Using equation (4) and the trim functions obtained in (6) the following qLPV model is obtained through some basic algebraic manipulations

$$\begin{bmatrix} \dot{z} \\ \dot{w} - \dot{w}_{eq} \end{bmatrix} = \begin{bmatrix} 0 & A_{12}(\rho) \\ 0 & A_{22}(\rho) - \frac{\partial w_{eq}}{\partial z} |_{\rho} A_{12}(\rho) \end{bmatrix} \begin{bmatrix} z \\ w - w_{eq}(\rho) \end{bmatrix} + \begin{bmatrix} B_1(\rho) \\ B_2(\rho) - \frac{\partial w_{eq}}{\partial z} |_{\rho} B_1(\rho) \end{bmatrix} \begin{bmatrix} u - u_{eq}(\rho) \end{bmatrix} \tag{7}$$

The qLPV model represents the nonlinear system generated through an exact transformation. References[2,9,10,13] provide with more in-depth derivations and discussion of this approach.

The Function Substitution approach was proposed in reference[7] for qLPV systems with nonlinearities in the control input (recall that qLPV systems must be linear with respect to the non-scheduling states and control inputs). In reference[7] a transformation of the nonlinear input parameter was performed to obtain a linear input. The system was then casted into an qLPV model where the real input was computed through a scheduled inverse of the nonlinear input. A similar case arises in references[2,11] which requires the design of synthetic control inputs to enable the qLPV modeling of the F-16 aircraft.

The procedure to obtain the quasi-LPV model is as follows. Define functions of the following form

$$\eta_z = z - z_{eq} \tag{8}$$

$$\eta_w = w - w_{eq} \tag{9}$$

$$\eta_u = u - u_{eq} \tag{10}$$

where z_{eq} is a chosen trim condition and w_{eq}, and u_{eq} the corresponding trim values for the non-scheduling states and control inputs. Substituting equations ($8 \to 10$) into equation (4) and rearranging terms

$$\begin{bmatrix} \dot{\eta}_z - \dot{z}_{eq} \\ \dot{\eta}_w - \dot{w}_{eq} \end{bmatrix} = \begin{bmatrix} A_{11}(\eta_z + z_{eq}) & A_{12}(\eta_z + z_{eq}) \\ A_{21}(\eta_z + z_{eq}) & A_{22}(\eta_z + z_{eq}) \end{bmatrix} \begin{bmatrix} \eta_z \\ \eta_w \end{bmatrix} + \begin{bmatrix} B_1(\eta_z + z_{eq}) \\ B_2(\eta_z + z_{eq}) \end{bmatrix} [\eta_u] + \mathcal{F}(\eta_z, w_{eq}, u_{eq}) \quad (11)$$

where

$$\mathcal{F}(\eta_z, w_{eq}, u_{eq}) = \begin{bmatrix} A_{11}(\eta_z + z_{eq}) & A_{12}(\eta_z + z_{eq}) \\ A_{21}(\eta_z + z_{eq}) & A_{22}(\eta_z + z_{eq}) \end{bmatrix} \begin{bmatrix} z_{eq} \\ w_{eq} \end{bmatrix} + \begin{bmatrix} B_1(\eta_z + z_{eq}) \\ B_2(\eta_z + z_{eq}) \end{bmatrix} [u_{eq}] + \begin{bmatrix} k_1(\eta_z + z_{eq}) \\ k_2(\eta_z + z_{eq}) \end{bmatrix} \quad (12)$$

The objective is to decompose $\mathcal{F}(\eta_z, w_{eq}, u_{eq})$ into functions linear in $\eta_z \in \mathcal{R}^{n_z}$ and then substitute the result back into equation (11).

$$\mathcal{F}(\eta_z, w_{eq}, u_{eq}) = f_1(z)\,\eta_{z1} + f_2(z)\,\eta_{z2} + \ldots + f_n(z)\,\eta_{zn} \quad (13)$$

The decomposition can be posed as an optimization problem

min ε

subject to

$$\mathcal{F}(z_1, z_2 \ldots, z_n) = f_1(z)\,\eta_{z_1} + f_2(z)\,\eta_{z_2} + \ldots + f_n(z)\,\eta_{z_n}$$
$$|f_i(z) - f_{i_{eq}}| \leq \Gamma\,|f_{i_{eq}}| + \varepsilon \quad \text{for } i = 1, 2, \ldots, n$$

where $\mathcal{F}$, from equation (12) at a fixed trim condition, and Γ, a measure of the change in the derivative, are known parameters. The objective is to minimize ε, variations in $f_i(z)$, over all possible η_z. The unknown functions, $f_i(x)$, will be used to obtain the desired decomposition once evaluated at the chosen trim position. This can be resolved as well as a linear program, see reference[2] .

Substituting the decomposed function $\mathcal{F}$, equation (13), in equation (11) the qLPV model is obtained

$$\begin{bmatrix} \dot{z} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} A_{11}(z) + f1(z) & A_{12}(z) \\ A_{21}(z) + f2(z) & A_{22}(z) \end{bmatrix} \begin{bmatrix} z - z_{eq} \\ w - w_{eq} \end{bmatrix} + \begin{bmatrix} B_1(z) \\ B_2(z) \end{bmatrix} [u - u_{eq}] \quad (14)$$

where z_{eq}, w_{eq}, and u_{eq}, the previously chosen trim condition, are fixed.

The main disadvantage of this approach is a lack of theoretical validation. There remain several open questions, among them the importance of the trim point chosen and the effects of the type of decomposition in the subsequent LPV controller synthesis stage.

4 Boeing 747-100/200

The aircraft model used in this work is the Boeing 747 series 100/200. This aircraft was chosen since its wide array of characteristics (leading and trailing edge flaps, spoilers, variety of control surfaces, four fan jet engines, ...) make of it the perfect representative for any of the commercial airplanes flying today, and thus an ideal test bed to prove the versatility of the LPV modeling and design techniques. The Boeing 747 is an intercontinental wide-body transport with four fan jet engines designed to operate from international airports. Some of its performance characteristics are a range of 6,000 nautical miles, a cruising speed greater than 965 kilometers per hour and a design ceiling of 13,716 meters.

The body-axes longitudinal motion of the Boeing 747, not including flexible effects, can be described by the following differential equations (assuming no wind)

$$\dot{\alpha} = \frac{[-F_x \cdot s_\alpha + F_z \cdot c_\alpha]}{m \cdot V_{TAS}} + q \quad (15)$$

$$\dot{q} = c_7 \cdot M_y \quad (16)$$

$$\dot{\theta} = q \quad (17)$$

$$\dot{V_{TAS}} = \frac{1}{m} \cdot [F_x \cdot c_\alpha + F_z \cdot s_\alpha] \quad (18)$$

$$\dot{h_e} = V_{TAS} \cdot c_\alpha \cdot s_\theta - V_{TAS} \cdot s_\alpha \cdot c_\theta = V_{TAS} \cdot sin\gamma \quad (19)$$

Longitudinal control is performed through a movable horizontal stabilizer with four elevator segments. Pitch trim is provided by the horizontal stabilizer, σ, and under normal operation the inboard and outboard elevators move together, $\delta_E = \delta_{E_I} = \delta_{E_O}$.

The body-axes aerodynamic forces and moments are given by

$$F_x = -\bar{q}S \cdot [C_D \cdot c_\alpha - C_L \cdot s_\alpha] + \sum_{i=1,4} Tn_i - mg \cdot s_\theta \quad (20)$$

$$F_z = -\bar{q}S \cdot [C_D \cdot s_\alpha + C_L \cdot c_\alpha] - 0.0436 \cdot \sum_{i=1,4} Tn_i + mg \cdot c_\theta \quad (21)$$

$$M_y = \bar{q}S\bar{c} \cdot \{C_m - \frac{1}{\bar{c}}[(C_D \cdot s_\alpha + C_L \cdot c_\alpha) \cdot \bar{x}_{cg} - (C_D \cdot c_\alpha - C_L \cdot s_\alpha) \cdot \bar{z}_{cg}] + \frac{\bar{c}\dot{\alpha}}{V_{TAS}}[C_{m_{\dot{\alpha}}} - \frac{\bar{x}_{cg}}{\bar{c}} \cdot C_{L_{\dot{\alpha}}} \cdot c_\alpha]\} + \sum_{i=1,4} Tn_i \cdot zeng_i \quad (22)$$

The aerodynamic data for the Boeing 747-100/200 was obtained from references[14,15] . Since the full set of aerodynamic coefficients was deemed too complex, an analytical study of the importance of each stability derivative with respect to the nominal value of the given aerodynamic coefficient was done. Then, open-loop time simulation comparisons between this first reduced model and the complete set were performed to ascertain the validity of the final reduced set. In Figure 8, the time reponses of both aerodynamic systems to a 1.2 degree step input of the elevator applied at t=15 seconds are given. The difference in the angle of attack is due to software constraints, i.e. the trim subroutine used the angle of attack as an independent variable. The details of this reduction can be obtained in[13] where the six aerodynamic coefficients were studied. The reduced aerodynamic coefficients for the longitudinal motion are given below

$$C_L = C_{L_{basic}}(\alpha_w, M) + \frac{dC_L}{dq}(he, M)\cdot\frac{q\bar{c}}{2V_{TAS}}\cdot[1.45 - 1.8x_{c.g.}] + K_\alpha(\alpha_w)\cdot[\frac{dC_L}{d\delta_{E_I}}(he, M) + \frac{dC_L}{d\delta_{E_O}}(he, M)]\cdot\delta_E \quad (23)$$

$$C_D = C_{D_{Mach}}(M, C_L^*) \quad (24)$$

$$C_m = C_{m_{basic}}(\alpha_w, M) + \frac{dC_{m0.25}}{dq}(he, M)\cdot\frac{q\bar{c}}{2V_{TAS}} + K_\alpha(\alpha_w)\cdot\frac{dC_{m0.25}}{d\sigma}(he, M)\cdot\sigma_{F.R.L.} + K_\alpha(\alpha_w)\cdot[\frac{dC_{m0.25}}{d\delta_{E_I}}(he, M) + \frac{dC_{m0.25}}{d\delta_{E_O}}(he, M)]\cdot\delta_E \quad (25)$$

The lift coefficient, C_L, depends only on the effects of the pitch rate, the elevators (which enter linearly), and the basic component, $C_{L_{basic}}$. The pitching moment aerodynamic coefficient, C_m, has the same dependencies as the lift coefficient plus a term to account for the effect of the horizontal stabilizer, which also enters linearly. The drag coefficient, C_D, has only one term accounting for the effect of the Mach number. Unfortunately, looking into the dependencies of this stability derivative, $C_{D_{Mach}}(M, C_L^*)$, it is seen that it depends on Mach number and the first two terms in the lift coefficient equation.

$$C_L^* = C_{L_{basic}} + \frac{dC_L}{dq}\cdot\frac{q\bar{c}}{2V_{TAS}}\cdot[1.45 - 1.8x_{c.g.}] \quad (26)$$

It is remarked the dependency on the pitch rate state in equation (26). This will pose a problem since one of the requirements for LPV modeling is to have the nonlinear equations linear in the pair $[w\ u]$ where w is the vector formed by the non-scheduling states and u the control input vector.

5 Software

The software used in this project to simulate and analyze the behavior of the Boeing 747 is an enhanced version of Flight Lab 747, FTLAB747, which is now capable of operating in a MATLABv5.3.1 environment. FTLAb747 program and its predecessor, Delft University Aircraft Simulation and Analysis Tool, DASMAT, were developed by Delft University of Technology, The Netherlands (see references[16–18]). DASMAT has been used for many years as a learning tool at Delft University simulating and analyzing among other aircraft a twin-engined business jet, the Cessna Citation 500. FTLAB747 was developed specifically to study the EL AL Israel Airlines (ELY) 1862 crash accident on October 4^{th} 1992 near Amsterdam, and is a particularisation of DASMAT registering changes pertaining the Boeing 747 and accounting for flight failures.

Both programs are based in Matlab/Simulink, and offer a wide array of simulation and analysis tools, i.e. trim, linearisation, simulation, and flight visualization. The structure of the programs is modular and exploit the advantage of decoupling general from aircraft specific dynamics, allowing easy implementation of other aircraft.

The trim subroutine returns the control inputs and aircraft and engine states that correspond to a flight condition where the linear and angular accelerations are zero. It provides six different options for trimming of the aircraft. They are straight-and-level, push-over/pullover, level turn, thrust-stabilized turn, beta trim and specific power turn (see references[16, 19]). Each condition requires different constraints and independent variables. The independent variables are primarily the main flight controls and the aircraft/engine states, and they are the variables tuned by the trimming subroutine. The operating point constraint, specified by the user, is in both programs the true airspeed (or Mach number) and the altitude. For most of the trimming options the angle of attack, α, and the sideslip angle, β are used as independent variables. When reducing the aerodynamic coefficients model, see Section 4, the time simulations and the corresponding trim conditions were carried out with FTLAB747v5.3. Since the only variables the user can specify are the altitude and true airspeed, the time responses for the different models at the same flight condition may start at slightly different angles of attack and sideslip.

The scheduling variables to be used in the qLPV model are angle of attack, true airspeed and altitude. For the qLPV models a new trimming subroutine using the MATLAB command *fminsearch* is used since it is required to specify the angle of attack, the true airspeed and the altitude of the aircraft, see next section.

6 Quasi-LPV models

The scheduling variables to be used in the quasi-LPV model are angle of attack, true airspeed and altitude, thus $z = [\ \alpha\ \ V_{TAS}\ \ h_e\ ]^T$, $w = [\ q\ \ \theta\ ]^T$ and $u = [\ \delta_e\ \ \sigma\ \ Tn\ ]^T$. The angle of attack, α, and true airspeed, V_{TAS}, are a common choice for this class of motion. It is decided to include the altitude as a scheduling parameter due to the dependence of most of the stability derivatives on Mach and altitude. Since the scheduling variables are all states of the system, this will result in a quasi-LPV model.

It was highlighted before, Section 4, that the longitudinal nonlinear EoM for the Boeing 747 do have some non-scheduling states entering the system in a nonlinear manner, i.e. the pitch angle enters the A, and f matrices through trigonometric functions, and the pitch rate the f matrix through the drag coefficient. Therefore some transformations are needed to cast the nonlinear equations in the appropriate form (i.e. the class of nonlinear systems mentioned in Section 3, equation (4)).

In order to transform the nonlinear entries, a linearisation with respect to a trim value is performed.

$$cos\theta = cos\theta_{eq} - sin\theta_{eq} \cdot \nabla_\theta \tag{27}$$

$$sin\theta = sin\theta_{eq} + cos\theta_{eq} \cdot \nabla_\theta \tag{28}$$

where ∇_θ is the difference between the state and a trim point.

$$C_{D_{Mach}}(M, C_L^*) \equiv C_{D_{Mach}}(\alpha_w, q, M, h_e) \tag{29}$$

$$C_{D_{Mach}}(\alpha_w, q, M, h_e) \approx C_{D_{Mach}}(\alpha_w, q_{eq}, M, h_e) + \left.\frac{\partial C_{D_{Mach}}}{\partial q}\right|_{eq} \cdot (q - q_{eq}) \tag{30}$$

After these approximations the new dependencies of the equations of motion will be as required.[13]

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \\ \dot{V}_{TAS} \\ \dot{\theta} \\ \dot{he} \end{bmatrix} = A(\alpha, V_{TAS}, he, \theta_{eq}) \begin{bmatrix} \alpha \\ q \\ V_{TAS} \\ \nabla_\theta \\ he \end{bmatrix} + B(\alpha, V_{TAS}, he) \begin{bmatrix} \delta_e \\ \sigma \\ Tn \end{bmatrix} + f(\alpha, V_{TAS}, he, \theta_{eq}, q_{eq}) \tag{31}$$

Next, it is necessary to study the feasible trim map since the State Transformation and the Jacobian Linearisation are dependent on trim functions and/or trim points. In the Aerospace Industry is common to trim by zeroing out the aerodynamics forces and moments (20 $\rightarrow$ 22). It is easy to verify that $q_{eq} = 0$ and $\theta_{eq} = \alpha$ (straight-level-flight). Figure 2 shows the acceptable trim region in terms of the angle of attack and the true airspeed at a given altitude, in this case 7000 meters.

State Transformation quasi-LPV model

From the qLPV-ready equation (31) it is straight forward to obtain the quasi-LPV model for the longitudinal motion of the Boeing 747 using the State Transformation approach. Using the trim subroutine it is possible to calculate trim points in terms of the scheduling vectors although there is no assurance that feasible trim values exist for a particular flight condition. Since $q_{eq}(\rho) = 0$, and $\theta_{eq}(\rho) = \alpha$ for any value of $\rho = (\alpha,\ V_{TAS},\ h_e\ )$ their derivatives are

$$\begin{bmatrix} \frac{\partial q_{eq}(\rho)}{\partial \rho} \\ \frac{\partial \theta_{eq}(\rho)}{\partial \rho} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{32}$$

Hence, the quasi-LPV system for the longitudinal motion of the Boeing 747 using State transformation, equation (7), is a basic reshuffling of the terms in the nonlinear equation (4). Figure 3 show the open-loop time responses of the nonlinear system and the quasi-LPV models to a step elevator deflection of -0.5 degrees applied after 2 seconds. The flight condition is the same for all the time responses: angle of attack of 2.29 degrees, true airspeed of 203 meters/second, and an altitude of 7000 meters. It is observed that the behavior matches almost perfectly. Results for different control inputs deflections showed also a good matching as far as the magnitude of the deflections was such that the airplane will be flying inside the trim region[13] . Figure 4 shows the behavior of the quasi-LPV model when a strong deflection is used, i.e. the control input is a $\pm$ 5 degrees doublet from time 2 $\rightarrow$ 10 second and 20 $\rightarrow$ 28 seconds. In this simulation whenever the trim values were not acceptable (i.e. no feasible trim achieved) they were just fed to the model to be able to plot the time responses.

A note about the implementation of the State Transformation and the Jacobian. Due to the meager trim-map instead of calculating a family of plants at several trim points and then interpolate, we were forced to update continuously the plants. This proved a mixed blessing since the common errors associated with interpolation did not occur, but on the other hand, the computational time required to simulate the model was exceedingly long.

Jacobian Linearisation quasi-LPV model

The Jacobian quasi-LPV model is straight forward. By performing linearisations with respect to a generic trim point it is possible to obtain a state-space description of the model where all the dependencies are in terms of the scheduling variables. The same trim subroutine as in the State Transformation can be used to obtain the trim values for the other states and control inputs. Matlab subroutines are used to obtain and update the state-space system and then calculate the new states. In reference[13] the quasi-LPV model is given in detail together with the complete derivation for one of the states.

In Figure 5, a control perturbation of the elevator surface equal to -0.5 degrees from $2 \leq$ time ≤ 30 seconds is applied. The simulation time is 30 seconds. The quasi-LPV model and the nonlinear model time responses match each other almost perfectly. Figure 6, shows the time responses to the same strong input as in the State Transformation to investigate the effect of a large deviation from the trim region. It is observed that the quasi-LPV model is not able to follow the nonlinear response. Even more reveling is the fact that at time 20 seconds (i.e. when the -5 degrees deflection is input) the simulation stopped producing an error.

Function Substitution quasi-LPV model

The last approach used to obtain the quasi-LPV model is the Function Substitution. As mentioned before in Section 3, this approach models the nonlinear system around a unique trim point. The effect the different choices of this equilibrium point have on the LPV control synthesis is not yet fully known, time responses of models obtained at different trim points showed not marked difference, see Section 7. It is expected that the functions obtained from the decomposition using a particular trim point are smooth in order to avoid problems during the control synthesis due to wide variations in the scheduling parameters. The class of nonlinear systems used in the other two approaches, equation (31), is also the departure point for this method.

The equilibrium point selected is the same as before. Using an optimization routine based on the problem posed in Section 3, the function $\mathcal{F}$ is decomposed into three functions, see equation (33). The function $\mathcal{F}$ is evaluated for the entire flight envelope so that the functions obtained through the decomposition are basically determined by minimizing all possible variations in the envelope.

$$\mathcal{F}(z) = \begin{bmatrix} f_{11}(z) & f_{12}(z) & f_{13}(z) \\ f_{21}(z) & f_{22}(z) & f_{33}(z) \\ f_{31}(z) & f_{32}(z) & f_{33}(z) \end{bmatrix} \begin{bmatrix} (\alpha - \alpha_{eq}) \\ (V_{TAS} - V_{TAS_{eq}}) \\ (he - he_{eq}) \end{bmatrix} \tag{33}$$

The Function Substitution quasi-LPV model is obtained by substituting equation (12) by the above decomposed function (33) in the nonlinear model obtained from equations (31) and ($8 \rightarrow 10$).

Figure (7) shows the response to the aforementioned deflection of the elevator. After 20 seconds simulation there are small differences in the time responses but these are minimal and is obvious that the model is able to follow the nonlinear response.

The last figure, Figure 8, shows the advantage of this method with respect to the others. The deflection is again the strong doublet in the horizontal stabilizer. The quasi-LPV model is able to follow the nonlinear time responses very well. There are some differences in the true airspeed and altitude but again small in magnitude. The advantage comes from the fact that in this method it is required to calculate only one trim point around which the model is developed, while in the Jacobian and State Transformation the model is only valid in the region defined by the trim points. In practice this means that for small trim maps the other two quasi-LPV models might not be able to cover the entire flight envelope.

7 Comparisons and Conclusions

In this paper three quasi-LPV models for the Boeing 747 longitudinal axes have been presented. Each approach presents different advantages as well as disadvantages. To help in comparing the different qLPV models an index performance that measures the relative and absolute errors of the qLPV and nonlinear states responses is used, see reference[2] .

$$J = \sum_{i=1}^{n_z} J_i = \sum_{i=1}^{n_z} \frac{1}{t_f} \int_{t=0}^{t_f} \frac{(y_i(t) - \tilde{y}_i(t))^2}{S_i} \tag{34}$$

where $y_i(t)$ is the output for one of the nonlinear states, $\tilde{y}_i(t)$ is the output for the corresponding quasi-LPV state, t_f is the final simulation time, n_z is the number of states, and S_i is a pre-defined scaling factor for each state, $(\alpha, V_{TAS}, he, \theta, q)$, used to normalize them, see equation (35). The performance index J provides an indication about the size of the deviation of the quasi-LPV model from the nonlinear model, and thus a comparative measure of the three quasi-LPV models. The subscript “nl” in equation (35) indicates the value of the state for the nonlinear model.

$$S_i = [(\frac{\pi}{180})^2 \quad (V_{TAS_{nl}})^2 \quad (he_{nl})^2 \quad (\theta_{nl})^2 \quad (\frac{\pi}{180})^2] \tag{35}$$

The results for open-loop time simulations are given in Table (3). All the simulations are performed with respect to the same equilibrium point ($\alpha = 2.29$ deg, $V_{TAS} = 203$ m/s, and $h_e = 7000$ m) and using the same small control input disturbances, Table 1.

deflection 1	δ_e	$\begin{cases} -0.5 \text{ deg} & t \geq 2 \text{ sec} \\ 0 \text{ deg} & \text{else} \end{cases}$
deflection 2	σ	$\begin{cases} +0.5 \text{ deg} & 2 \leq t \geq 5 \text{ sec} \\ -0.5 \text{ deg} & 12 \leq t \geq 15 \text{ sec} \\ 0 \text{ deg} & \text{else} \end{cases}$
deflection 3	T_n	$\begin{cases} 40,000 \text{ Newton} & t \geq 2 \text{ sec} \\ 0 \text{ Newton} & \text{else} \end{cases}$

Table 1 Control Disturbances used for the Function Substitution qLPV models.

The indexes J1, and J5 are absolute errors with respect to the angle of attack and pitch rate states. The other indexes J2, J3, and J4 are relative errors with respect to the true airspeed, altitude, and pitch angle states respectively. The last column in Tables 3 and 4 is the average of the performance indexes. From Table 3, it is observed that the State and Jacobian quasi-LPV models are almost equal. This was expected from the graphs of the time simulations. This is explained by the continuous update of the quasi-LPV model performed in the Jacobian and State Transformation approaches. The function Substitution quasi-LPV model yields a difference two orders of magnitude larger than the other two quasi-LPV models but still on the order of 10^{-3} for the first and third deflections. For the second deflection the Function Substitution quasi-LPV model performs slightly better than the other two models except for J4. The magnitude of J4 is the result of a difference in sign in the responses which does not happen in the other models (i.e. the qLPV response becomes negative while the nonlinear stays positive for a short period of time).

The results shown in Table 3 might mislead into believing that the Jacobian and State Transformation quasi-LPV models are superior to the Function Substitution model for the Boeing 747. While for the trim condition and control inputs selected this is true, in the sense that the time responses of the quasi-LPV and nonlinear model are very close, it is necessary to account for the other characteristics of the models (i.e. modeling difficulty, advantages, drawbacks ...). Even more, when a large deflection was used only the Function Substitution quasi-LPV model was able to follow the nonlinear time response (see Figures 4, 6, and 8).

In terms of the difficulty of obtaining the quasi-LPV models is noted that once the nonlinear equations are transformed into the adequate format, equation (4), all the quasi-LPV models are relatively easy to obtain. The Jacobian approach presented an additional difficulty due to the number of partial derivatives involved in the modeling (this can be avoided by simply using one of the many availables trimming routines to find a family of LTI plants). The State Transformation in the Boeing 747 case was almost a simple re-shuffling of the terms in equation (4). The Function Substitution involved an optimization routine aimed at decomposing the nonlinear function given by equation (12). In our case and since the optimization software has been previously developed in collaboration with the University of California-Berkely this was a straight forward process (it was only necessary to adapt the software to the Boeing 747 specific characteristics).

Summarizing the advantages of each model it is observed that the Jacobian Linearisation approach is the most widespread methodology and it has a proven theoretical base. It is therefore easy to understand and to learn how to apply it to parameter varying systems. The State transformation provides an exact LPV model of the nonlinear system since it uses states transformations. The Function Substitution requires only one trim point around which to obtain the LPV model. All of them allow the use of LPV control synthesis techniques. The LPV control synthesis naturally fits into the gain-scheduling framework while providing assurances about the stability and robustness of the system.

The Jacobian LPV approach is a first order approximation of the system, and generally it is not possible to capture the transient behavior of the nonlinear model. The State Transformation model depends on the existence of continuously differentiable trim functions for the non-scheduling states and the inputs, $w_{eq}(\rho)$ and $u_{eq}(\rho)$. Unfortunately there are no assurances of their existence. Both approaches, the Jacobian and State Transformation, are restricted by the feasible trim map obtained. If this region is small, as it is in the present case, the simulation can be hampered by the limited range in which the model can exists. Also, in the case the LPV model is obtained by a family of plants evaluated at different trim points, this is true for both Jacobian and State LPV models, the required interpolations needed to obtain the behavior of the model at a different point result in further approximations. In this project and due to the meager trim map it was decided to calculate the state-space models at each step. This resulted in time responses close to the nonlinear behavior, although the computational time increased prohibitively. The Function Substitution suffers from a void of theoretical explanations. There are several open questions about the influence the choice of different trim points has on the LPV control synthesis, and about the smoothness of the decomposed functions (i.e. it is assumed that wide variations will negatively affect the LPV control synthesis). Using the index performance presented before, equations (34) and (35), models obtained using the Function Substitution approach for different trim points were compared. Table 2 shows the trim points around which the Function Substitution quasi-LPV models are obtained. In Table 4 the index performance results for the open-loop time simulation of those models with the same deflections as before is presented.

Model	alpha, deg	V_{TAS}, m/s	h_e, m
Model 1	2.29	203	7000
Model 2	4.5	185	9000
Model 3	5.5	195	10000

Table 2 Trim points used in Function Substitution quasi-LPV modeling.

From Table 4 it can be concluded that all the Function Substitution models regardless of the trim point selected result in similar open-loop time responses with respect to the nonlinear response. Of course, it will be necessary to extend the study of this relative dependency on the trim point to the LPV control synthesis to adequately calibrate its importance.

Since the final goal of developing quasi-LPV models is to enable the use of LPV control synthesis which might be easier to apply at trim and non-trim points in the flight envelope, it can be concluded that the model obtained by the Function Substitution will provide the best chances of successfully synthesizing an LPV controller.

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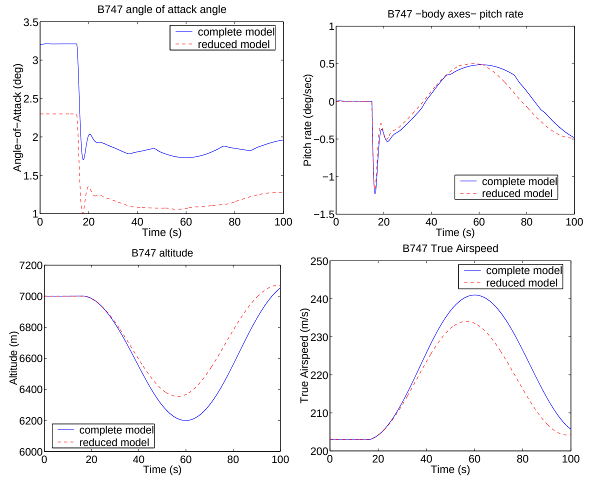

Fig. 1 Open-loop time response for the complete and reduced aerodynamic models.

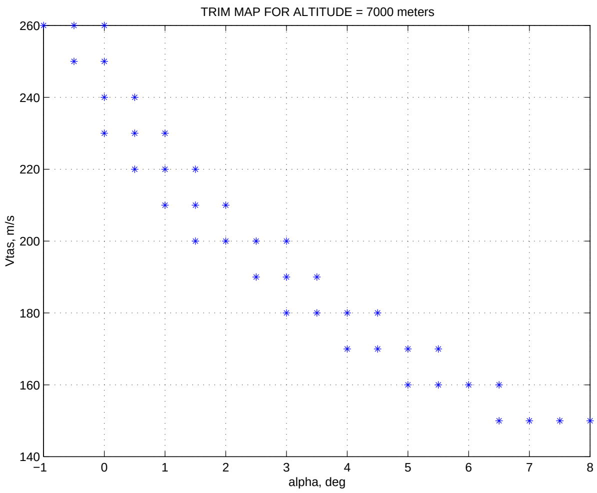

Fig. 2 Trim map at 7000 meters.

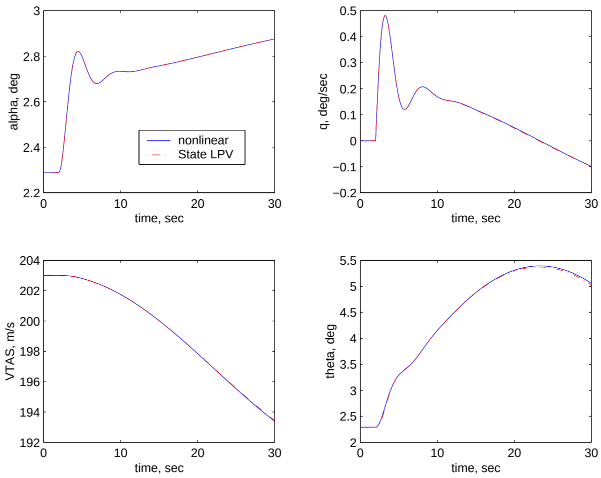

Fig. 3 State Quasi-LPV time response to an elevator deflection.

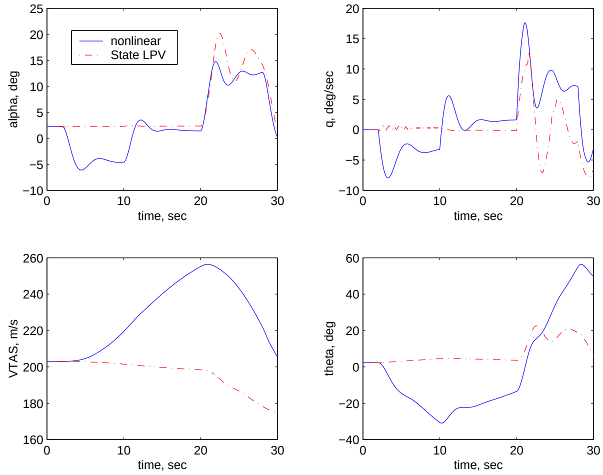

Fig. 4 State Quasi-LPV time response to a strong horizontal deflection.

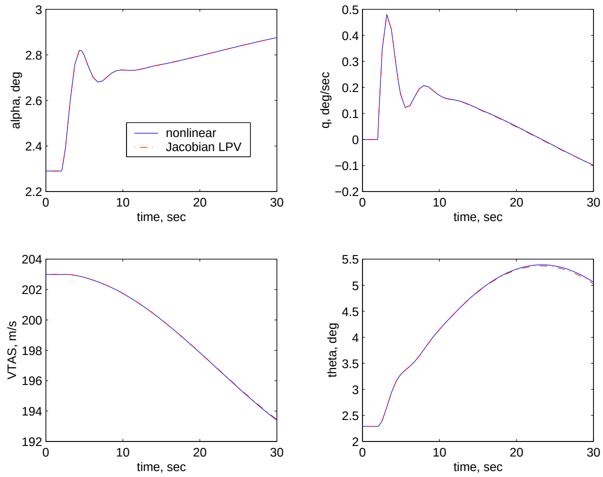

Fig. 5 Jacobian Quasi-LPV time response to an elevator deflection.

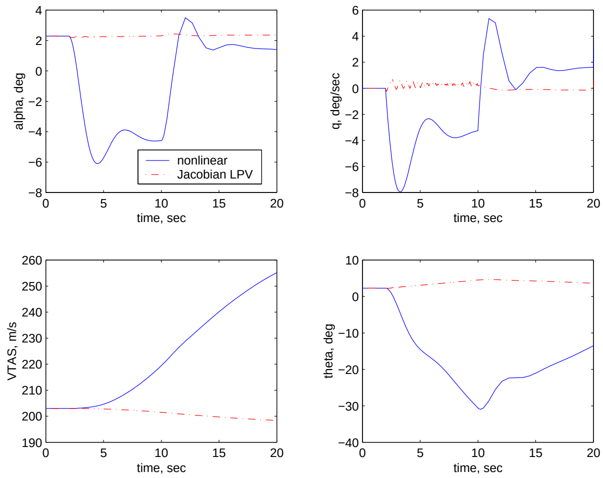

Fig. 6 Jacobian Quasi-LPV time response to a strong horizontal stabilizer deflection.

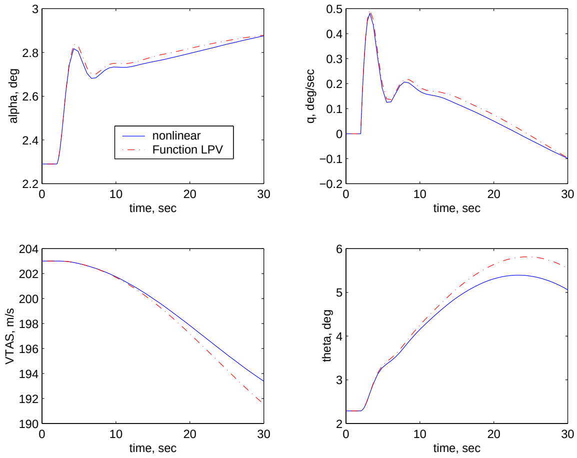

Fig. 7 Function Quasi-LPV time response to an elevator deflection.

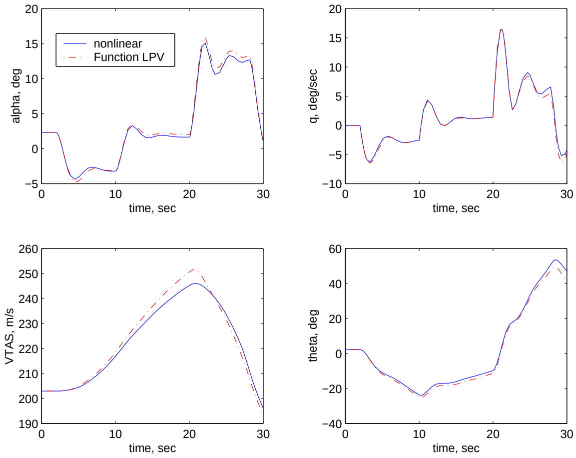

Fig. 8 Function Quasi-LPV time response to a strong horizontal stabilizer deflection.

State quasi-LPV	**J1**	**J2**	**J3**	**J4**	**J5**	**Σ**
Deflection 1	1.0795e-07	9.8247e-09	2.9887e-09	2.5076e-05	1.4440e-06	2.6641e-05
Deflection 2	7.0387e-03	1.0120e-05	4.1601e-06	2.7424e-02	1.2020e-02	4.6497e-02
Deflection 3	1.0087e-06	1.5456e-08	5.7213e-09	1.9477e-05	1.2392e-05	3.2899e-05
Jacobian quasi-LPV	**J1**	**J2**	**J3**	**J4**	**J5**	**Σ**
Deflection 1	1.1952e-07	9.5862e-09	2.8941e-09	8.0550e-06	1.5193e-06	9.7063e-06
Deflection 2	6.1719e-03	1.0168e-05	4.1533e-06	2.6711e-02	1.0574e-02	4.3471e-02
Deflection 3	1.0086e-06	1.5553e-08	5.7542e-09	1.5636e-05	1.2403e-05	2.9069e-05
Function quasi-LPV	**J1**	**J2**	**J3**	**J4**	**J5**	**Σ**
Deflection 1	3.2049e-04	1.5406e-05	1.3988e-06	2.9087e-03	3.3306e-04	3.5790e-03
Deflection 2	2.3678e-03	4.4575e-06	9.5467e-07	1.8523e-01	2.2607e-03	1.8987e-01
Deflection 3	5.0306e-04	1.2545e-06	4.6080e-07	4.9259e-04	1.7947e-04	1.1768e-03

Table 3 Performance index results.

Model 1	**J1**	**J2**	**J3**	**J4**	**J5**	**Σ**
Deflection 1	3.2049e-04	1.5406e-05	1.3988e-06	2.9087e-03	3.3306e-04	3.5790e-03
Deflection 2	2.3678e-03	4.4575e-06	9.5467e-07	1.8523e-01	2.2607e-03	1.8987e-01
Deflection 3	5.0306e-04	1.2545e-06	4.6080e-07	4.9259e-04	1.7947e-04	1.1768e-03
Model 2	**J1**	**J2**	**J3**	**J4**	**J5**	**Σ**
Deflection 1	6.4105e-02	1.1647e-04	4.9051e-06	2.1448e-02	4.4403e-03	9.0115e-02
Deflection 2	7.6890e-04	4.2747e-07	2.7247e-08	1.7194e-04	7.1921e-04	1.6605e-03
Deflection 3	6.5196e-04	2.9204e-07	9.7373e-09	1.0934e-04	2.0958e-04	9.7118e-04
Model 3	**J1**	**J2**	**J3**	**J4**	**J5**	**Σ**
Deflection 1	2.1129e-02	8.9422e-06	8.9531e-07	2.2385e-03	7.6397e-03	3.1017e-02
Deflection 2	4.4024e-02	2.6300e-05	2.1834e-06	7.3881e-03	2.6012e-02	7.7453e-02
Deflection 3	7.2824e-04	5.6977e-08	2.2911e-08	1.2250e-04	1.6434e-04	1.0152e-03

Table 4 Comparison Function Substitution models.