Flight Dynamics Application of a New Symbolic Matrix Order-Reduction Algorithm

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In the field of control, the basis of state-space theory is the representation of a system in a matrix-like form and its subsequent manipulation for synthesis and analysis. The state-space model represents any system, given by nonlinear first-order ordinary differential equations (ODE), as a 2×2 block matrix $\begin{bmatrix} \dot{x} \\ y \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}$ using state variables which map the input-output behaviour of the system. The (two) inputs of this compact matrix representation are the state vector x and the nonlinear system input vector u while the (two) outputs are the first-order derivative of the state vector $\dot{x}$ and the output vector y of the nonlinear system. In general, a first-order numerical approximation to the nonlinear system is performed to obtain the state-space system matrices A, B, C and D. This standard numerical modelling approach introduces approximation errors and can be considered as a 'black-box' modelling method from the perspective of nonlinear system analysis.

A paradigm shift in control-oriented modelling occurred twenty years ago with the introduction of modern robust theory and its associated modelling framework, the linear fractional transformation (LFT) representation. The LFT framework is an extension of the state-space modelling approach which allows the representation of a nonlinear system as a linear feedback interconnection of two matrices: a nominal matrix which represents the known and linear part of the system, and a structured (diagonal) matrix containing the uncertainties and/or nonlinearities of the system. The order of an LFT is defined as the dimension of the structured matrix, which is determined by the total sum of the nonlinear and uncertain parameters (including their repetitions) that compose the structured matrix. The LFT order and the number/type of the different parameters are important considerations that limit the applicability of current controller design and analysis techniques [1].

Although in the computer algebra field symbolic techniques are very well-known for matrix and polynomial manipulations, they have only recently been exploited in the field of control for LFT model order reduction [2,3] and only even more recently software implementations have appeared [4,5]. The order-reduction LFT technique proposed and implemented in [5] represents a novel order-reduction algorithm for multivariate symbolic polynomial matrices which stems from those in [2,3]. In this paper, an application of this order-reduction LFT algorithm to flight dynamic modelling is given. The representation of the nonlinear longitudinal dynamics of a Boeing 747-100/200 aircraft in the required symbolic state-space matrix M for the application of the LFT order-reduction algorithm is based on the nonlinear symbolic modelling framework from reference [6].

The basic structure of the LFT order-reduction algorithm [5] is a combination of a modified structured-tree decomposition [2] and of the Horner factorization approach [3]. Additionally, tree-height reduction techniques from symbolic algebra [7] and specially defined metrics are used to improve the performance of the algorithm and to decide on the optimal ordering of the variables and the direction of the decomposition. The objective of the algorithm is to decompose a multi-variate symbolic polynomial matrix into an equivalent LFT representation of reduced order. The algorithm is divided into three main routines: information management, nested factorization and sum decomposition. The information manager calculates a set of metrics and used them to decide on the optimal matrix decomposition. The main metrics are the "presence" degree σ, the factor order fac, the reduction order red, and the "possible" reduction order red_{pos}. The "presence" degree σ

is defined as the number of times, including powers, a variable appears in a symbolic expression. It can be viewed as a polynomial, or matrix, extension of the relative degree of a monomial. The factor order of a variable is the maximum power to which the variable can be factored out from an expression. The reduction order *red* is the largest reduction in the “presence” degree of an expression achievable through factorization of a variable. The last metric, red_{pos}, is used in the event that some coefficients in the expression do not contain the specified variable. It corresponds to the achievable reduction order assuming these *trouble* coefficients or monomials have been removed.

The nested factorization stage includes finding the multivariate Horner form for each matrix coefficient (if they are polynomials) and performing an affine factorization of the resulting Horner matrix. The previous metrics are used to estimate the optimal ordering of the symbolic parameters list so that the Horner form of the multivariate polynomials using this list yields the largest matrix reduction order. The symbolic tree-height reduction technique ‘substitution’ is used to transform the Horner polynomial matrix into a monomial (or monomial *mono* plus constant, *ct*) matrix where the nested polynomials from the Horner form are substituted by dummy variables, $p = ct + poly \cdot mono = ct + dummy \cdot mono$. The resulting monomial matrix M_{mono} is then affine-factorized: $M_{mono} = M_{cte} + L_{sym} M_{sym} R_{sym}$ where M_{cte} contains the constant terms, L_{sym} and R_{sym} are identity matrices with those symbolic parameters that could be factorized along the diagonal (L for those factorizable along the rows and R for those along the columns), and M_{sym} contains all the rest of the symbolic parameters.

The sum decomposition stage aims to optimally decompose M_{sym} into two matrices, $M_{sym} = M_{symA} + M_{symB}$, so that further nested factorizations of these two matrices yield additional order reduction gains. The optimal decomposition is achieved by back-substituting the dummy variables in M_{sym}, expanding the matrix, and obtaining a decomposition list. This parameter list provides the required information on how to distribute the M_{sym} coefficients across both sum-decomposed matrices based on the optimal ordering of the symbolic parameters and their resulting reduction and “possible” reduction orders. If M_{sym} has polynomial coefficients, it is transformed into an augmented monomial matrix before the sum decomposition, which results in this case in similarly augmented M_{symA} and M_{symB}. The augmented matrix is obtained by expanding the polynomial summands across columns so that the resulting matrix gives all the combinations between the different summands and monomials in each column of M_{sym}, e.g. $\begin{bmatrix} m1+m2 \\ m3+m4 \end{bmatrix} = \begin{bmatrix} m1 & m1 & m2 & m2 \\ m3 & m4 & m3 & m4 \end{bmatrix}$. The augmented M_{symA} and M_{symB} are collapsed after the decomposition back to their proper dimensions, those of M, before continuing with the algorithm. The algorithm iterates using M_{symA} as the new matrix until it can not be decomposed further, in which case it continues with each successive M_{symA} and M_{symB} until all are completely decomposed.

In this paper, the modelling framework from reference [6] is used in the nonlinear aircraft longitudinal system from [8] to show the applicability of the algorithm given in [5]. The modelling approach first represents the nonlinear system as a symbolic nonlinear state-space system in the standard 2×2 block matrix format M (i.e. a multivariate symbolic matrix) and subsequently utilizes the order-reduction algorithm to obtain a close to minimal LFT representation with a structured matrix that contains all the symbolic parameters in diagonal format. The nonlinear symbolic LFT approach has the advantage of allowing a more transparent trade-off between modelling simplicity and system fidelity. It also allows for a more straight forward study of the nonlinearities effects, while providing an easy way to correct erroneous simplifications or mis-judgements with minimal effort and impact on the overall modelling process.

The body-axes longitudinal motion of the Boeing 747, not including flexible modes or wind effects, can be described by the following ordinary differential equations:

$$\dot{\alpha} = \frac{[-F_x\, s_\alpha + F_z\, c_\alpha]}{m\, V_{TAS}} + q \qquad \dot{q} = c_7\, M_y \tag{1}$$

$$\dot{\theta} = q \qquad \dot{V_{TAS}} = \frac{1}{m}\,[F_x\, c_\alpha + F_z\, s_\alpha] \tag{2}$$

$$\dot{h_e} = V_{TAS}\, c_\alpha\, s_\theta - V_{TAS}\, s_\alpha\, c_\theta \tag{3}$$

The states of the system are angle of attack, α (rad), pitch rate, q (rad/s), pitch angle, θ (rad), true airspeed, V_{tas} (m/s) and altitude, h_e (m). Longitudinal control is performed through a movable horizontal stabilizer, δ_{stab} (rad), four elevator segments (assumed to move in unison, δ_e (rad)) and the thrust from the four engines, Tn_i. The engine and control surfaces actuation are contained within the aerodynamic forces F_x, F_z and moments M_y (not shown due to space constraints). Pitch trim is provided by the horizontal stabilizer.

Five symbolic constants are declared together with seventeen different symbolic parameters. The symbolic constants are physical terms assumed known and fixed regardless of flight condition or time variations (e.g. wing chord and gravity). The different symbolic parameters include all the physical terms that are varying or uncertain, stability derivatives (given in the form of tabular data) and nonlinear terms in the nonlinear functions. The resulting nonlinear symbolic state-space form for the nonlinear system has a total "presence" degree of 156 and it is given in an initial factorized form due to the physical characteristics of the original ODE system.

Three approaches are used to transform the nonlinear symbolic state-space into a nonlinear symbolic LFT: 'direct', 'tree decomposition' and the proposed 'Horner-tree decomposition'. The 'direct' approach uses the command *lfrs* from ONERA's toolbox [4] and involves simple manipulations but, except for basic symbolic expressions, results in LFTs with the highest order. The 'tree decomposition' approach [2] is also implemented in [4] under the command *symtreed*. A first release of the implementation of the third approach, which can be obtained from the first author, uses the command *symHtree* to obtain the reduced order LFT.

Two additional minimization techniques, also available in [4], are used on the resulting final LFTs. The first minimization technique implements a 1-D reduction approach whereby one of the parameters is selected while the others are assumed to be normal inputs and outputs of the system. The second minimization technique applies an N-D approach that considers simultaneously all the parameters in the structured matrix. Table 1 shows the results of the application of the order-reduction LFT and minimization techniques to the nonlinear symbolic matrix:

Algorithm	**no min**	**min 1D**	**min ND**
Direct	72	54	54
Structured-tree	55	46	45
Horner-tree code	51	44	43

Table 1: Aircraft LFT modelling results.

It is observed that as each of the tree decomposition approaches are applied a further reduction on the LFT order is obtained. The 'structured tree' decomposition achieves a reduction of 17 parameters (23 percent approximately) with respect to the 'direct' *un-minimized* LFT. The proposed approach results in a further reduction of 7 percent (4 fewer parameters) in the LFT order compared to the 'structured-tree' decomposition.

In order to apply robust control techniques using the previous LFT models, the symbolic parameters in the structured matrix must be normalized to unity. Normalization of the parameters has been recommended to be applied as the last step in the LFT modelling process [2,3,4,5] prior to the application of the control design and analysis techniques. This was proposed based on the perceived added complexity of the normalized system prior to the decomposition stage [2,3,4] and in order to keep the natural definition of the symbolic parameters for as long as possible [5]. The first assertion is clarified in the following analysis, see Table 2, which shows that it is not so much the normalization process that increases the system complexity and LFT order but the construction of the expression (whether it is factorized or fully expanded) and the type of decomposition scheme used.

For the 'direct' approach, it is noticed that the effect of normalization is nill when the model is already in compact form (see the 2nd and 4th models in Table 2) but that when applied to the expanded form (1st and 3rd models) it has a very dramatic effect on the LFT order for the *un-minimized* LFT process. Furthermore, the effects of normalization and expansion in this approach are reduced when the 1-D minimization technique is applied, and are negligible for the ND minimization. This shows that for the 'direct' LFT modelling approach the normalization introduces

		direct			Horner tree		
Exp	**Norm**	**no-min**	**min1**	**minND**	**no-min**	**min1**	**minND**
YES	YES	241	78	57	70	57	57
NO	YES	72	54	54	70	57	57
YES	NO	117	83	55	51	44	43
NO	NO	72	54	54	51	44	43

Table 2: LFT models: ‘expansion’ and ‘normalization’ effects on LFT order.

increased complexity only when the system is fully expanded. Hence, it is not the normalization per se but rather its combination with the construction of the system that is mainly responsible for the increase in complexity and LFT order.

For the ’Honer tree’ decomposition, the normalization process does make a difference regardless of the initial construct of the system. This is because this approach uses the technique ‘expansion’ within the algorithm prior to the system decomposition (so that in fact, the models being analyzed are the first and third, i.e. with and without normalization, since the second and fourth are the same as the previous two respectively).

The LFT N-D minimized models given in Table 1 were simulated in open and closed loop and compared to the nonlinear system with no noticeable differences in their responses since the reduction in the number of parameters is not associated with a loss of modelling fidelity (the plots are not included for lack of space).

In conclusion, in this extended abstract a flight dynamics application of a novel symbolic order-reduction algorithm for LFT modelling has been shown. The first important conclusion obtained is that the model-reduction step and the minimization techniques should be incorporated routinely into the LFT modelling process. Also, the reduction in the number of parameters is not associated with a loss of modelling fidelity since the reduction is accomplished by matrix manipulations.

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