Computing the (almost) exact value of μ with the **SMART** library of the **SMAC** toolbox

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http://w3.onera.fr/smac

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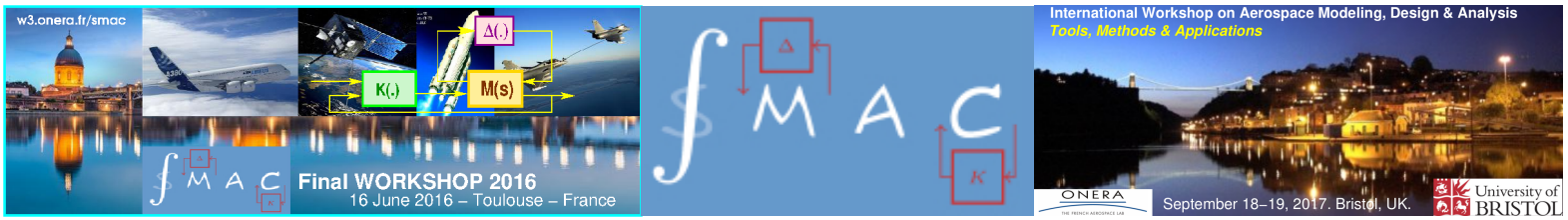

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Context and objective

Control laws design is usually based on a (linear) mathematical model, which significantly simplifies/alters reality.

Such a model is not a perfect representation of the real behavior of a physical system because of:

- high-frequency uncertainties (neglected dynamics),
- uncertainties on the parameters which characterize the system (*e.g.* mass, inertia, aerodynamic coefficients),
- time-varying parameters:
  - fast variations (*e.g.* mass of a launcher during atmospheric flight),
  - slow variations (*e.g.* mass, velocity, altitude of a transport aircraft),
- nonlinear phenomena:
  - aerodynamic phenomena at high angles,
  - actuators saturations,
  - transmission delays...

Robustness with respect to these phenomena must be ensured!

Classical industrial approach

Monte-Carlo simulations:

1. choose many random samples, each of them being composed of random operating points (e.g. mass configurations, CoG position...) and random inputs (e.g. pilot inputs, wind...),
2. perform a closed-loop simulation for each sample,
3. perform statistical analysis on the resulting output samples to get the probabilities that some stability, performance, loads, comfort... criteria are satisfied.

Advantage:

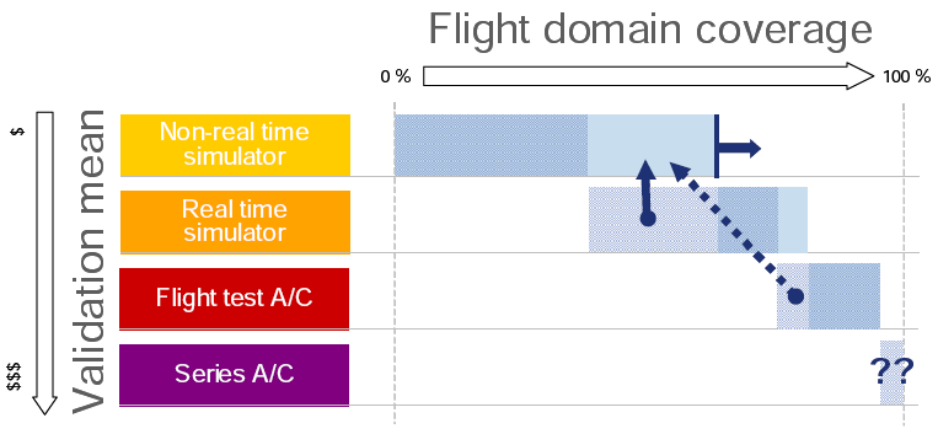
- easy to implement

Drawbacks:

- exponential-time approach $\Rightarrow$ high computational complexity
- statistical approach $\Rightarrow$ worst cases can be missed

New trend

Develop some **inexpensive tools** to determine quickly the most critical parametric configurations without performing extensive simulations.

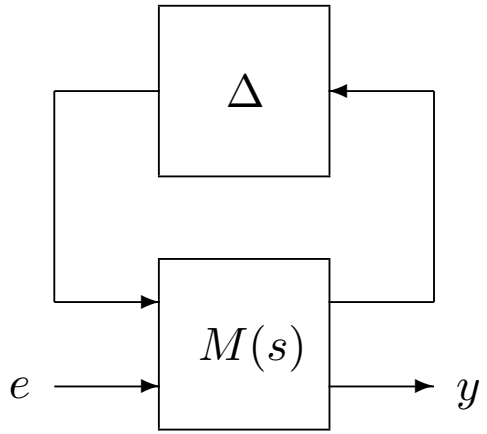


Many robustness analysis problems can be formulated as optimization problems, where it must be checked that a set of criteria lie within certain limits for all admissible system configurations.

Some techniques such as μ, IQC-based or Lyapunov-based analysis can be efficient alternatives to Monte-Carlo simulations.

Different analysis techniques

Preliminary task: transform the uncertain/time-varying/nonlinear model of the system into a Linear Fractional Representation, *e.g.* using the **GSS & APRICOT libraries** of the SMAC Toolbox.



Several system components can be isolated in the Δ block, such as:

- parametric / dynamic uncertainties
- time-varying / time-invariant parameters
- non-linearities (saturations, deadzones, sector non-linearities. . .)

Different analysis techniques

The selected analysis technique depends on the elements in Δ:

- time-invariant uncertainties: μ-analysis
  $\rightarrow$ **SMART library** (**S**kew **M**u **A**nalysis based **R**obustness **T**ools)
- uncertainties, time-varying parameters & non-linearities: IQC-based analysis
  $\rightarrow$ **IQC library** (**I**ntegral **Q**uadratic **C**onstraints)
  $\rightarrow$ **SeDuMi-based IQC solver**
- saturations, deadzones and sector non-linearities: Lyapunov-based analysis and nonsmooth multi-objective H_∞ optimization
  $\rightarrow$ **SAW library** (**S**aturated systems analysis and **A**nti-**W**indup design)
  $\rightarrow$ **OISTER library** (**O**ut to **I**n **S**aturation **T**ransformation **E**xtensions for **R**obustness)

All these tools are available on the SMAC website:
http://w3.onera.fr/smac

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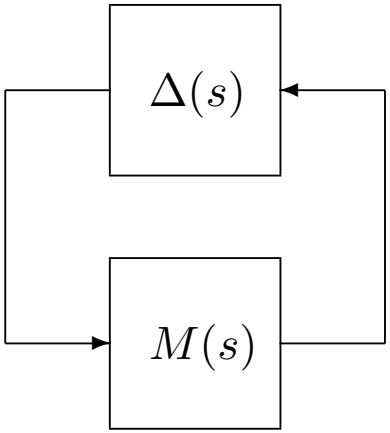
Problem statement

Stability/performance of LTI systems with time-invariant uncertainties.



$M(s)$ is a stable and proper real-rational transfer function $\Rightarrow$ nominal system.

$\Delta(s) = \text{diag}(\Delta_1(s), \ldots, \Delta_N(s))$ is a block-diagonal operator $\Rightarrow$ model uncertainties. $\Delta_i(s)$ can be:

- a time-invariant diagonal matrix $\Delta_i(s) = \delta_i I_{n_i}$, where δ_i is a real/complex parametric uncertainty,
- a stable and proper real-rational unstructured transfer function representing neglected dynamics.

$\mathbf{\Delta}$ is the set of all matrices with the same block-diagonal structure and the same nature (real or complex) as $\Delta(j\omega) \Rightarrow$ admissible uncertainties.

$k\mathcal{B}_{\mathbf{\Delta}} = \{\Delta \in \mathbf{\Delta} : \overline{\sigma}(\Delta) < k\} \Rightarrow$ maximum size of the uncertaintes.

Robust stability margin

Definition of the structured singular value

$$\mu_{\mathbf{\Delta}}(M(j\omega)) = \left[\min_{\Delta \in \mathbf{\Delta}} \{\overline{\sigma}(\Delta),\, \det(I - M(j\omega)\Delta) = 0\} \right]^{-1}$$

Small gain theorem for structured uncertainties

The interconnexion $M(s) - \Delta(s)$ is stable $\forall \Delta(s) \in k\mathcal{B}_{\mathbf{\Delta}}$ if and only if:

$$\sup_{\omega \in \mathbb{R}_+} \mu_{\mathbf{\Delta}}(M(j\omega)) \leq \frac{1}{k}$$

Definition of the robust stability margin

$$k_r = \left[\sup_{\omega \in \mathbb{R}_+} \mu_{\mathbf{\Delta}}(M(j\omega)) \right]^{-1}$$

- the interconnexion is stable $\forall \Delta(s)$ of size strictly less than k_r
- $\exists \Delta(s)$ of size k_r for which the interconnexion is unstable

Standard computational approach

The exact computation of the robust stability margin k_r is NP hard in the general case, so both lower and upper bounds are computed instead:

- a lower bound (*i.e.* a μ upper bound $\overline{\mu}_{\mathbf{\Delta}}$) provides a guaranteed but conservative value of k_r,
- an upper bound (*i.e.* a μ lower bound $\underline{\mu}_{\mathbf{\Delta}}$), usually associated to a worst-case parametric configuration, allows to quantity the conservatism of the lower bound.

But even computing these bounds is a challenging problem with an infinite number of frequency-domain constraints.

It is usually solved on a finite frequency grid $(\omega_i)_{i\in[1,M]}$ and an estimate of the robust stability margin is then obtained as:

$$\frac{1}{\max\limits_{i\in[1,M]}(\overline{\mu}_{\mathbf{\Delta}}(M(j\omega_i)))} \leq k_r \leq \frac{1}{\max\limits_{i\in[1,M]}(\underline{\mu}_{\mathbf{\Delta}}(M(j\omega_i)))}$$

Limitations of the standard approach

Problem

The grid must contain the most critical frequency point for which the maximal value of $\mu_{\mathbf{\Delta}}$ is reached.

If not:

- The upper bound on k_r can be very poor, notably for flexible systems, for which the function $\mu_{\mathbf{\Delta}}(M(j\omega))$ often exhibits very high and narrow peaks.
- Even worse, the lower bound on k_r can be over-evaluated, *i.e.* be larger than the real value of k_r.

Unfortunately, the aforementioned critical frequency is usually unknown!

In this context, the considered frequency grid must be sufficiently dense, which can lead to a prohibitive computational cost.

But even so, it is still possible to miss a critical frequency...

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Objectives

Main purposes of the SMART library

- implement state-of-the-art algorithms to compute the (almost) **exact value of μ** in (almost) all cases with a low computational time, even for large size problems addressed by control engineers.
- ensure that reliable results are obtained, *i.e.* that no critical frequency is missed and that the computed margins are guaranteed on the whole frequency range.
- solve a wide class of problems, including (skewed) robust stability margin, worst-case H_∞ performance level and worst-case gain/phase/modulus/delay margins.
- propose a user-friendly Matlab toolbox, which can be used both by researchers and by control engineers.

$\Rightarrow$ Go beyond the **Robust Control Toolbox**, which can prove useful but:

- only addresses a limited number of issues,
- does not propose any solution when the gap between the bounds on μ is large,
- usually considers a finite frequency grid instead of the whole frequency range.

An alternative approach

To achieve these goals, and compute both tight and reliable bounds on k_r, alternative methods are implemented in the **SMART library** [MSC 2011, MSC 2013]:

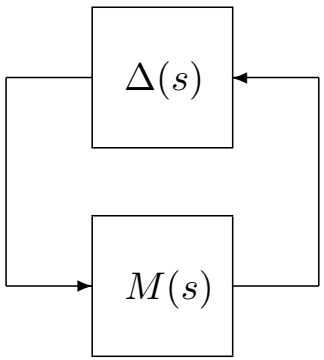
- An μ upper bound is first determined at some frequency. **A hamiltonian-based technique** is then applied to determine all frequency intervals on which this bound remains valid. This strategy is repeated and **a guaranteed lower bound on k_r** is finally obtained when the union of all intervals covers the whole frequency range.

- **A poles migration technique** is applied to compute the smallest possible perturbation $\tilde{\Delta}(s) \in \mathbf{\Delta}$ such that the interconnection between $M(s)$ and $\tilde{\Delta}(s)$ is unstable. A μ lower bound, and thus **a guaranteed upper bound on k_r**, is obtained. Frequency is an optimization parameter, which **allows to detect critical frequencies and usually leads to very accurate bounds**.

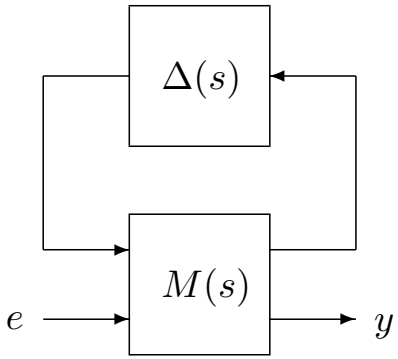
- **A branch & bound algorithm** is implemented, which **tightens the gap between the bounds on k_r to the desired accuracy at a reasonable computational cost**.

Addressed issues

Different kinds of problems can be solved with the **SMART library**:



1. **robust stability margin** - Compute the robust stability margin k_r for a given block structure $\boldsymbol{\Delta}$.
2. **skewed robust stability margin** - Compute the skewed robust stability margin for a given block structure $\boldsymbol{\Delta}$.



3. **worst-case H_∞ performance level** - Compute the highest value of the H_∞ norm of the transfer matrix from e to y when $\Delta(s)$ takes all possible values in $\mathcal{B}_{\boldsymbol{\Delta}}$ [MSC 2010].
4. **worst-case input-output margins** - Compute the worst-case gain, modulus, phase and delay margins, *i.e.* the highest value of the (real or complex) gain, the phase shift or the time delay that can be inserted between y and e without destabilizing the system when $\Delta(s)$ takes all possible values in $\mathcal{B}_{\boldsymbol{\Delta}}$ [CDC 2011].

About the Matlab code

The **SMART library** has been tested on systems with up to 100 states and 500 uncertainties (counting repetitions). It usually gives very accurate results, only requires a moderate computational effort, and has proved successful in many real-world applications.

- It can be used on all platforms (Linux, Mac and Windows).
- All routines run on Matlab R2010b or higher. They should run on older releases, but no validation has been performed.
- It is fully compatible with `gss`, `lfr` and `uss` objects.
- Each routine is documented and a complete on-line documentation is available.

The **full version** is freely available on the SMAC website
http://w3.onera.fr/smac/smart

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List of benchmarks

A large set of **36 challenging benchmarks** is considered:

- a few academic systems and many real-world applications
- purely real, mixed real/complex or purely complex uncertainties
- large number of states (from 2 up to 70)
- large number of uncertainties (from 1 up to 28), repeated or not, even highly repeated in some cases (size of Δ up to 507×507)
- both rigid & highly flexible models (aircraft, telescope mockup, satellite...)
- several fields of application (civilian & fighter aircraft, launcher, re-entry vehicle, satellite, telescope, helicopter, spacecraft, missile, hard disk drive, biochemical network, car, hydraulic servo system, spark ignition engine...)

32 benchmarks are available in the control literature. The other 4 ones have been developed by ONERA in cooperation with industrial partners.

k_r upper bound computation ($\Leftrightarrow$ μ lower bound)

All existing polynomial-time μ lower bound algorithms are applied to compute upper bounds on k_r for each benchmark [CEP 2015].

	Description	Uncertainties	Matlab code
1	Power algorithm	all	Robust Control Toolbox
2	Gain-based algorithm	all but complex	Robust Control Toolbox
3	Poles migration technique	all	Carsten Döll
4	Poles migration technique	real	SMART library
5	Poles migration technique	all	Mark Halton
6	Direct nonlinear optimization	all	Mark Halton
7	Direct nonsmooth optimization	all	Alberto Simoes
8	Geometrical approach	real	Jongrae Kim

Results for purely real problems (1-29)

Algorithm	Number of times the gap w.r.t. the best μ lower bound is			Mean gap w.r.t. the best μ lower bound	Mean CPU time
	$=0\%$	$\leq 5\%$	$\leq 25\%$		
1	6	9	13	44.10%	1.7 s
2	2	19	24	11.87%	27.1 s
3	5	16	22	18.81%	1.0 s
4	26	27	29	0.88%	0.9 s
5	22	26	26	8.95%	22.8 s
6 (g)	5	18	24	11.09%	448.8 s
6 (i)	9	20	23	15.72%	97.8 s
7 (g)	8	17	23	17.44%	1694.9 s
7 (i)	24	25	25	9.36%	124.3 s
8	0	8	17	24.38%	749.4 s

The most relevant algorithm is the **poles migration technique of Ferreres & Biannic** [CEP 2001] with both the highest accuracy and the lowest CPU time.

Improvement for purely real problems

Improving the poles migration technique is not a trivial issue.

Idea: combine several algorithms

1. algorithm 4 is executed first (most efficient technique in almost all cases),
2. algorithm 2 is then executed only for a few selected frequencies with previous results as initialization,
3. particle swarm optimization is finally applied with previous results as initialization.

This strategy is implemented in the **SMART library**.

Benchmark	Algorithm 4		Other algorithms			Combination	
	value	time	best value	time	algo	value	time
20	0.9380	0.5 s	0.9947	41.6 s	7	0.9947	5.5 s
26	0.9881	2.2 s	1.2134	184.9 s	7	1.2144	18.0 s
29	724.15	46.9 s	733.86	2864.8 s	2	753.10	580.0 s

The best lower bound is obtained for all benchmarks with this strategy.

Results for purely complex and mixed problems (30-36)

Only algorithms 1-3-5-6-7 can be applied.

Algorithm	Number of times the gap w.r.t. the best μ lower bound is			Mean gap w.r.t. the best μ lower bound	Mean CPU time
	$=0\%$	$\leq 5\%$	$\leq 25\%$		
1	4	7	7	0.10%	1.1 s
3	0	4	6	16.60%	0.9 s
5	2	2	3	57.12%	140.8 s
6 (g)	1	7	7	0.34%	2648.8 s
6 (i)	0	5	7	4.26%	874.2 s
7 (g)	4	4	6	10.63%	3972.6 s
7 (i)	2	5	6	7.91%	249.5 s

The most relevant algorithm is the **power algorithm of Young & Doyle** [TAC 1997] with the highest accuracy and almost the lowest CPU time.

Improvement for purely complex and mixed problems

Idea: better exploit the power algorithm

1. algorithm 1 is applied on a rough frequency grid (*e.g.* 20 frequency points)
2. the grid is gradually tightened around the peak frequencies until improvement in the μ lower bound becomes marginal

At each frequency, algorithm 1 is not only initialized with the best result obtained at the previous frequency, but also with one or more random values.

This strategy is implemented in the **SMART library**.

Benchmark	Algorithm 1		Other algorithms			Improved algo 1	
	value	time	best value	time	algo	value	time
33	0.4346	0.7 s	0.4362	1380.1 s	7	0.4362	0.6 s
34	0.9587	1.1 s	0.9604	565.8 s	7	0.9606	1.3 s
35	0.9910	1.7 s	0.9927	12.3 s	5	0.9927	2.1 s

The best lower bound is obtained for all benchmarks with this strategy.

k_r lower bound computation ($\Leftrightarrow$ μ upper bound)

What about conservatism, *i.e.* the gap w.r.t. the exact value of k_r?

A lower bound on k_r is computed for each of the 36 benchmarks using the **hamiltonian-based technique of Biannic et al** [IFAC 2005] **and Roos & Biannic** [ACC 2010] implemented in the **SMART library**.

The mean gap between the best k_r lower and upper bounds is:

- **12.71%** for purely real problems
- **0.39%** for purely complex and mixed real/complex problems

Is it the lower or the upper bound which is responsible for this gap?

Claim

The k_r upper bound almost always equals the exact value of k_r.

No proof, but true in many practical cases!

Branch & bound to reduce conservatism...

Purely complex and mixed problems:

- For 6 benchmarks out of 7, the gap is less than 0.01%.
- For the last one, a standard **branch & bound algorithm** allows to increase the lower bound on k_r until a gap less than 0.01% is obtained.

The exact value of k_r is obtained in all cases.

Purely real problems:

- For 19 benchmarks out of 29, the gap is less than 0.01%.
- For the 10 others, a standard **branch & bound algorithm** allows to increase the lower bound on k_r, leading to a gap less than 0.01% in 7 cases, and of 0.12%, 1.99% and 10.00% in the last 3 cases.

The mean value of the gap over all 29 benchmarks is now only **0.42%**.

The exact value of k_r is (almost) obtained in all except one case.

...at the price of an increase in computational time

Standard branch & bound algorithm

Cut the uncertainty domain in more and more subsets until the gap between

- the highest lower bound computed on all subsets and
- the highest upper bound computed on all subsets

becomes lower than a user-defined threshold η.

Branch & bound allows to reduce conservatism to an arbitrarily small value for systems with only real parametric uncertainties.

Problem

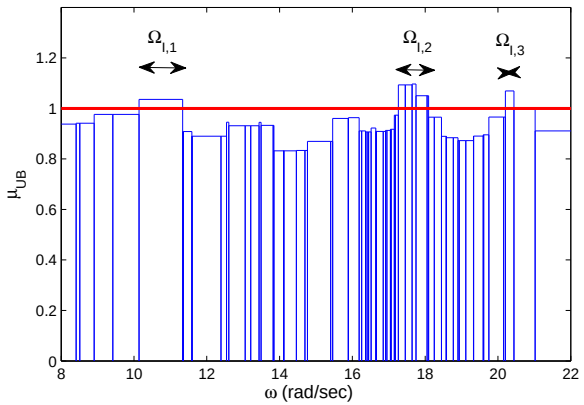
Computational complexity grows exponentially as a function of the number of uncertainties. Computing tight k_r lower bounds can be extremely long.

Towards a reduction of computational time

Improved branch & bound algorithm [MSC 2011]

Assume that for a given subset $\mathcal{D}_i$, stability can only be proved for frequencies $\omega \in \Omega_V \subset \mathbb{R}_+$:

- $\rightarrow$ $\mathcal{D}_i$ is partitioned,
- $\rightarrow$ for each subset of $\mathcal{D}_i$, the analysis is restricted to the frequency domain $\Omega_I = \mathbb{R}_+ \setminus \Omega_V$.



After a few steps, each analysis is restricted to narrow frequency intervals.

$\Rightarrow$ drastic reduction of computational cost, but still not sufficient...

Use of the μ-sensitivities

The μ-sensitivities aim at determining how much the μ upper bound is affected by a small variation of a single uncertainty in Δ.

They can be used to identify which uncertainties have the largest influence on the μ upper bound.

Idea

Cut the uncertainties with the highest μ-sensitivities only, so as to further improve the branch & bound algorithm.

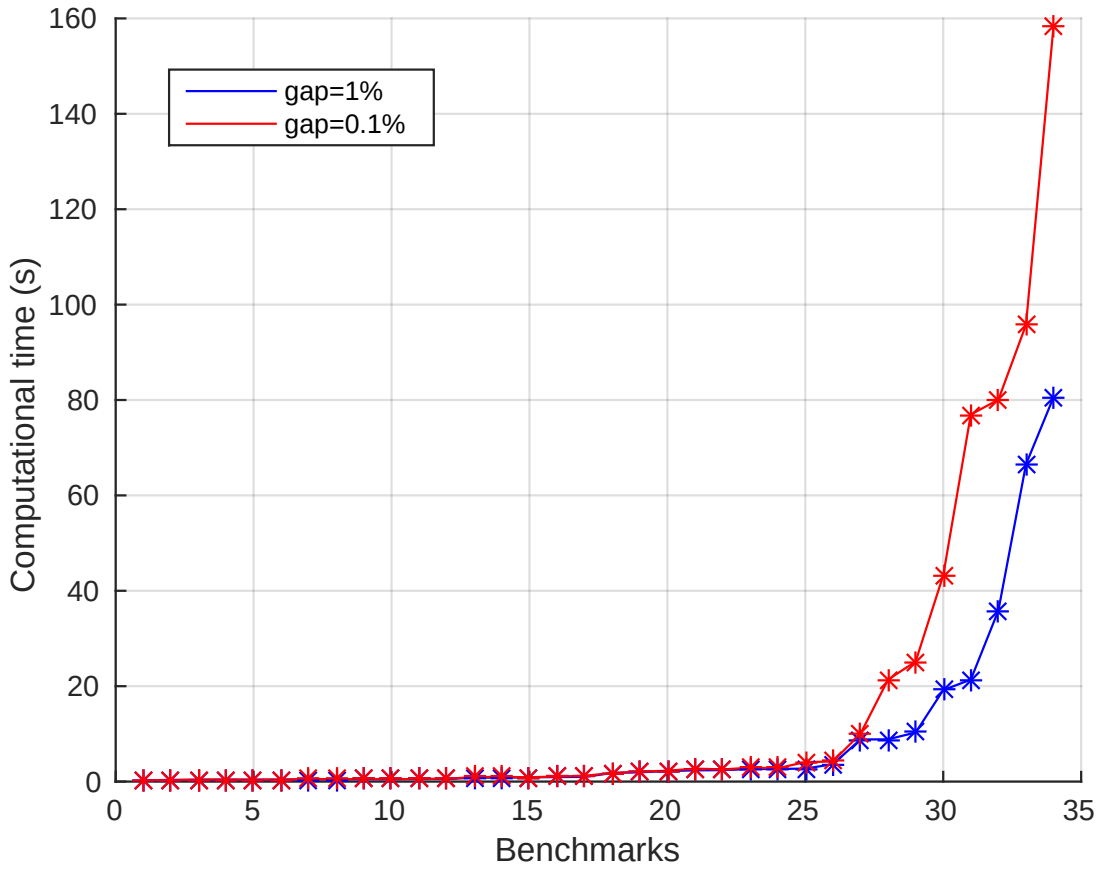
The number of selected uncertainties allows to handle the tradeoff between accuracy and computational complexity [ROCOND 2015].

The same accuracy is obtained with a much lower computational time.

This **branch & bound algorithm** is implemented in the **SMART library**.

Approaching the exact value of μ

For each benchmark, the computational time required to obtain a gap of either 1% or 0.1% between the μ upper and lower bounds (using the improved branch-and-bound algorithm if needed) is displayed.



Approaching the exact value of μ

For 26 benchmarks out of 36, the computational time is negligible:

- less than 5 s for a gap of 0.1%

For 8 benchmarks, the computational time is low:

- between 8 s and 80 s for a gap of 1%
- between 10 s and 158 s for a gap of 0.1%

Only 2 benchmarks require a larger computational effort:

- launcher **(30 states and 45×45 Δ matrix)**: 9 min 30 s for a gap of 1%, much longer for a gap of 0.1%
- biochemical network **(7 states and 507×507 Δ matrix)**: 22 min 30 s for a gap of 10%, 3 h 04 min for a gap of 5%, 14 h 54 min for a gap of 1%, much longer for a gap of 0.1%

With few exceptions, computing the exact value of μ is no longer an issue.

Conclusion

The **SMART library** of the SMAC Toolbox implements the most efficient μ-analysis based algorithms developed at ONERA during the past 15 years.

It allows to compute:

- the (skewed) structured singular value,
- the (skewed) robust stability margin,
- the worst-case H_∞ performance level,
- the worst-case gain, modulus, phase and delay margins.

It can be applied to high-order systems with numerous uncertainties.

It has been validated on many realistic benchmarks, and the exact value of μ is obtained in almost all cases with a low computational time.

The **full version** is available on the SMAC website:
http://w3.onera.fr/smac/smart

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μ lower bound

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μ upper bound

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